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I WORKINPS PAPER

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Maurizio Franzini

A Labor Market Sorting Model of Scarring and Hysteresis

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A Labor Market Sorting Model of Scarring and Hysteresis*

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Abstract

Recessions displace workers and worsen their career prospects. On the other hand, downturns might spur reallocation of workers towards more efficient firms. Understanding which forces dominate is of crucial importance to assess the aggregate impact of business cycles. We build a search model with worker-firm heterogeneity and aggregate risk, in which workers' human capital accumulation depends on the quality of firms in their match. The framework allows to account for how recessions impact workers to firms sorting, and to inspect how workers' skills and firms' distributions jointly evolve with business cycles. We estimate the model on administrative data and show that persistent negative effects on the productivity of worker-firm matches dominate cleansing effects, with distortions in sorting and human capital accumulation accounting for approximately 60% of cumulative output losses. The model is then used to offer a rationale for the increased length of recessions and their heterogeneous welfare effects across age, income and human capital distributions.

Keywords: Economic hysteresis, Business cycle fluctuations, Human capital accumulation, Labor market sorting, Labor market scarring

JEL codes: J24, J63, E24, E32

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Sommario*

Le recessioni hanno effetti avversi sui lavoratori, aumentando la disoccupazione e peggiorando le prospettive di carriera. Allo stesso tempo, queste fasi di declino economico portano alla riallocazione dei lavoratori tra le imprese, con possibili guadagni di efficienza per l'economia. È dunque naturale chiedersi se le recessioni portino a isteresi, ossia a effetti negativi persistenti sulla produttività, oppure se prevalgano a lungo termine i potenziali effetti benefici dati dal fallimento delle attività improduttive e dalla conseguente riallocazione delle risorse.

Per rispondere a tale domanda, sviluppiamo un modello strutturale del mercato del lavoro che incorpora l'eterogeneità tra le imprese e i lavoratori, l'accumulo di capitale umano per i lavoratori, dipendente dalla produttività delle imprese ove essi sono impiegati, e una fonte di rischio aggregato che determini un ciclo economico. Stimiamo il modello utilizzando dati amministrativi italiani su datori di lavoro e dipendenti e dati di bilancio delle imprese per gli ultimi 25 anni.

Il modello stimato è in grado di fornire una rappresentazione realistica delle dinamiche salariali delle carriere lavorative, delle separazioni, delle transizioni da un lavoro all'altro e della distribuzione dei salari. I risultati della stima del modello mostrano come gli effetti distorsivi delle recessioni, rispetto all'allocazione dei lavoratori nelle aziende, siano dominanti rispetto a supposti effetti positivi di riallocazione.

Il nostro risultato principale è mostrare che, a seguito di uno shock recessivo, l'impatto negativo sulle carriere dei lavoratori riflette non solo un peggioramento delle prospettive d'impiego, ma anche una riduzione della capacità produttiva dell'economia. Molteplici canali contribuiscono a questo risultato. In primo luogo, durante un evento recessivo molti lavoratori perdono il lavoro. In secondo luogo, la crescita salariale diminuisce per tutti i lavoratori ancora impiegati. In terzo luogo, la probabilità che i lavoratori trovino un nuovo lavoro è inferiore.

Grazie al modello stimato, siamo in grado di quantificare l'importanza delle distorsioni nella allocazione dei lavoratori verso le aziende e nell'accumulo di capitale umano suddividendo la perdita cumulativa di PIL delle recessioni in differenti canali. A tre anni da una recessione circa il 48% delle perdite deriva da una peggiore allocazione di lavoratori verso le imprese. La diminuzione dell'accumulo di capitale umano rappresenta un ulteriore 17%, mentre il rimanente è attribuibile agli effetti meccanici dati dalla diminuzione dell'impiego. Le distorsioni di allocazione dei lavoratori e di capitale umano spiegano complessivamente circa due terzi della perdita cumulata di produzione. Ciò implica che le recessioni abbiano un effetto depressivo sull'economia, portando a un deterioramento persistente della qualità dei lavori, che determina una perdita di produzione che l'economia non riassorbe mai completamente.

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1 Introduction

Recessions impact workers by forcing displacements and worsening career prospects. At the same time, economic downturns lead to the reallocation of workers across firms, with possible efficiency gains for the aggregate economy. It is then natural to ask whether recessions result in hysteresis, that is persistent negative effects on the productivity of workers and firms, or if they foster long-run efficiency-enhancing cleansing effects.

Understanding which forces determine the relative importance of hysteretic or cleansing effects requires modeling workers heterogeneity in skill and career progression, their matching to differently productive firms, and how these distributions jointly evolve with business cycles. To this end, we develop a directed search model that incorporates heterogeneity among firms and workers, endogenous match-dependent human capital accumulation, an overlapping generations structure, on-the-job search, and aggregate risk. Our focus is on a novel channel that links on-the-job human capital accumulation to labor market sorting—i.e., the extent to which more productive workers are assigned to more productive firms—and their joint interaction with output fluctuations. We estimate the model using administrative matched employer-employee and firm balance sheet data from Italy. The estimated model is able to provide a realistic portrayal of life cycle earnings profiles, career progression, separations, job-to-job transitions, and workers' cross-sectional earnings dynamics.

We show that persistent negative effects on the productivity of worker-firm matches dominate cleansing effects after recessions. The observed persistence is a consequence of recessions impairing sorting and distorting human capital accumulation paths. Our findings are consistent with empirical evidence of initial cleansing effects, followed by a persistent loss in productivity (Haltiwanger, Hyatt, McEntarfer and Staiger, 2022). Quantitatively, we estimate that almost two-thirds of cumulative losses in output after three years from the onset of a recession can be attributed to distortions of sorting of workers to firms and losses in human capital. The collapse of the job ladder in recessions, together with the resulting losses in human capital accumulation, generate long-term costs to output that are absent in other models of sorting in the business cycle, as Lise and Robin (2017) and Baley, Figueiredo and Ulbricht (2022).

A key ingredient of our framework is the presence of match-dependent human capital accumulation. An expanding body of literature highlights the significant role of firm quality in shaping workers' human capital accumulation and skill development (see Herkenhoff, Lise, Menzio and Phillips 2018; Lise and Postel-Vinay 2020; Arellano-Bover and Saltiel 2022; Koffi, Ozkan, Salgado and Weissler 2023).[1] We allow the production of

[1]Other studies highlighting the relevance of firms characteristics for earning dynamics over the life cycle and across educational levels are Engbom and Moser (2017), De La Roca and Puga (2017), Mion, Opromolla and Ottaviano (2022), Deming (2023).

human capital to be contingent upon the match between workers and firms, so that workers hired by more productive firms experience a faster accumulation of human capital.[2] Taking these factors into account drastically changes the interpretations of cleansing effects at the onset of recessions. The exit of less productive firms, usually seen as cleansing, impacts younger and less productive workers, with displacements producing a lasting impact on their careers (see also Huckfeldt 2022, Jarosch 2023). Short-run cleansing effects can therefore be detrimental to long-run growth, because displacements reduce human capital accumulation by keeping some workers stuck at lower rungs of the job ladder. In order to characterize the interaction between firm pay policies and workers' search incentives, we model the division of match surplus within the worker-firm relationship through a fully state-contingent wage protocol. Profit-maximizing firms strategically design a range of contracts to attract and retain the most productive workers. Workers cannot credibly commit to abstain from searching on the job, so contracts will take their search incentives into account. As workers are risk averse, there is also demand for insurance against fluctuations in match value, as in Holmstrom (1983), Thomas and Worral (1988), Balke and Lamadon (2022), and Souchier (2022). The state-contingent contract features front-loading of firms' profits and back-loading of workers' compensation, in order to maximize firm retention while partially insuring workers against idiosyncratic and aggregate shocks. Model simulations replicate well the within- and between-firm wage growth over the life cycle. Job-to-job flows are the key determinant of wage growth, as workers leave less productive firms for more productive ones, which can post better-paying contracts. Younger workers, who are less productive but more mobile, get higher wage increases in their jobs as firms try to increase retention. Less productive workers of all ages, who fail to climb the ladder, move in and out of unemployment largely by applying to less productive vacancies.

Further assumptions are needed in order to capture the role of cyclical displacements. The insurance provision from firms to workers implies that the wage schedule delivered by contracts is non-decreasing while the match has positive value, endogenously generating downward wage rigidity. Following a negative shock some matches might become unprofitable for firms. As we assume that contracts cannot be renegotiated ex-post, shocks might lead to inefficient separations even though both parties have full information about the contract provisions ex-ante.[3] Given these modeling ingredients, the model manages to closely match patterns of endogenous separations due to firm layoffs in the data across age, worker, and firm productivity. Displacements are more frequent in less productive firms, where it is more common for

[2] We also allow that, if high human capital workers are matched with very low productivity firms, their human capital can potentially *deteriorate* while on the job.

[3] A sizeable literature identifies substantial resistance to nominal or real wage cuts, and argues that layoffs are a more likely response to negative productivity shocks (Altonji and Devereux 1999; Agell and Lundborg 2003; Grigsby, Hurst and Yildirmaz 2021).

younger and less productive workers to be employed. Less productive firms are also disproportionately more likely to lay off workers in recessions. As a result, output and unemployment volatilities in the model match their empirical counterparts, a challenging result for standard search and matching models (see Shimer 2005). In addition, the pattern of wage growth and career dynamics following displacement reproduces a host of results from the empirical labor literature on scarring effects on workers careers (Jacobson, LaLonde and Sullivan 1993, Schwandt and von Wachter 2019, Schmieder, von Wachter and Heining 2022, Bertheau, Acabbi, Barcelo, Gulyas, Lombardi and Saggio 2023).

Despite the richness of its ingredients, two main features ensure the tractability of our framework. First, we prove that workers can map the productivity of each firm to the value of the posted vacancy, and their optimal search strategy is unique and monotonously increasing in their type. This effectively reduces the number of states of the problem and excludes the presence of multiple optima in workers' search strategies. Workers' search strategies are influenced by factors such as the level of human capital and firm productivity, effectively capturing the interplay between these elements in shaping workers' career trajectories. Second, the choice of modeling search as directed and the assumption of free entry allow us to prove that the model features a block recursive equilibrium (Menzio and Shi, 2010). The model can thus be solved without the requirement of keeping track of the agents' distributions as states of the optimization.

Bringing the model to the data allows us to characterize how aggregate shocks transmit in the economy, altering labor market sorting and workers' human capital accumulation, and to quantify the importance of worker-firm sorting and human capital accumulation channels for aggregate output losses.

Our main result is to show that, following a recessionary shock, the negative impact on workers' careers reflects not only a worsening of workers' outside options, but also an endogenous reduction in the overall productive capacity of the economy. Multiple channels contribute to this outcome. First, throughout a recessionary event many workers are displaced, and need to restart climbing the job ladder from the bottom. Second, wage growth decreases for all surviving matches, as workers' outside options deteriorate. Third, the probability of workers to move across jobs if employed, or towards any job if unemployed, is lower. The driving force behind these effects is that the job ladder flattens for *all* workers, because of a decrease in job openings and consequent lower job finding probabilities. We point out that the reduction in job openings is particularly pronounced for higher rungs of the ladder, as more productive vacancies are costlier to open and exhibit greater cyclicality. A flatter job ladder impairs workers' productivity growth and their upward mobility. The pro-cyclicality of high-quality jobs highlighted by Moscarini and Postel-Vinay (2016) is fully reproduced in our model. Workers match up the ladder with firms of lower quality, form less productive matches and thus accumulate less human capital. Worse worker-firm sorting thus amplifies and prolongs the negative effects of recessions on earnings and aggregate output observed in (Barlevy, 2002), leading to hysteresis.

With the estimated model at hand, we are able to quantify the relevance of the distortions in sorting and human capital accumulation by decomposing the cumulative output loss of recessions into four channels. We show that three years after a negative productivity shock approximately 48% of losses stems from a worsened sorting between workers and firms: distortions in workers' job finding probabilities with respect to normal times (*search* sorting channel) account for approximately 35%, whereas the remaining 13% is accounted for by the decrease in firm quality in matches formed throughout the recessionary and recovery periods (*firm quality* sorting channel). The decrease in human capital accumulation accounts for an additional 17% of cumulative output losses (*human capital* channel), whereas the remaining unexplained part is attributed to mechanical displacement effects (*displacement* channel). Taken together, distortions in sorting and human capital channels explain around two-thirds of the cumulative loss in output. The outcome of this exercise implies that recessions have a sullying effect on the economy: distortions in sorting and human capital accumulation lead to a persistent deterioration of matches' quality after a recession, which determines a loss in output that the economy never fully reabsorbs.

Furthermore, we estimate the welfare cost and incidence of business cycles. The welfare cost of business cycle is the utility amount agents would be willing to give away in order to eliminate aggregate risk. The model estimates this cost to be on average two orders of magnitude above what traditionally estimated by Lucas (1987) and broadly in line with more recent work (Barlevy, 2004). The incidence of these costs varies markedly by income and age. We find them to be roughly U-shaped in income, with the greatest costs at the extremes of the income distribution. For low–skill workers, which tend to have lower wages, costs are driven by displacement effects. For higher–skill workers, on the other hand, the distortions in sorting and human capital accumulation matter more than the immediate displacement effect. Business cycle costs also differ by age, with workers above forty years old suffering more than their younger counterparts. These workers have a harder time obtaining re-employment after displacement, and can be left with few options beyond early retirement.[4]

Moreover, the ability of the model to replicate within match pay dynamics allows us to relate firm-level labor share dynamics, roughly representing bargaining power of workers within the match, to aggregate output fluctuations. As observed in Dupraz, Nakamura

[4]These results for welfare echo and extend the results for earnings found by Heathcote, Perri and Violante (2020) and Doniger (2023), who find that recessions amplify income inequality and tend to have immediate stronger effects for lower income workers.

and Steinsson (2022), we show that the severity of recession and length of recoveries are state-dependent, and specifically positively related to the average and the skewness of the firm labor share distribution. These findings support the idea that the lack of labor market fluidity and dynamism, by favoring the buildup of labor share within matches, increases firms' riskiness and diminishes resilience in the labor market (Engbom, 2021).[5]

Finally, we discuss how the increase in the role of the sorting channel over time has affected the length of recoveries from recessions throughout business cycles. The model can be used to provide an explanation for the secular increase in the length of recessions across all developed economies, as observed for instance in Fukui, Nakamura and Steinsson (2023). We show that the *entirety* of the post 1990 increase in recession lengths can be attributed to a heightened importance of on-the-job learning for high skilled workers and population ageing.

Related literature. Our work takes inspiration from two strands of empirical research. We relate to the empirical literature analyzing the impact of business cycles on workers' earnings, exploring how recessions impact workers both via worse career prospects and by forcing displacements (Kahn 2010; Oreopoulos, Von Wachter and Heisz 2012; Schmieder et al. 2022); we also relate to the literature analyzing the effects of recessions on productivity, debating whether recessions tend to have long-lasting negative effects on the productivity distribution (Barlevy 2002; Haltiwanger et al. 2022) or, by freeing up resources for more efficient uses they generate cleansing effects (Davis, Haltiwanger and Schuh 1996). We use the insights from these empirical literatures to calibrate our model and benchmark our untargeted results in the estimation, and then use the model to analyze the productivity long run dynamics after a recessionary period.

Our model of the labor market builds on the directed search models developed in Menzio and Shi (2010), Menzio, Telyukova and Visschers (2016). We also adopt a wage setting protocol taken from the literature on dynamic contracts as in Thomas and Worral (1988) and, more recently, Balke and Lamadon (2022). In both cases, we extend their framework to explicitly account for two-sided heteronegeneity and on-the-job human capital accumulation to study how business cycles interact with workers' decisions, firms' optimal retention policies, and influence labor market sorting.

The centrality of worker-firm sorting and on-the-job learning in busyness cycle dynamics is closely related to Lise and Robin (2017), Baley et al. (2022), Herkenhoff, Phillips and Cohen-Cole (2023), Carrillo-Tudela and Visschers (2023), Carrillo-Tudela, Visschers and Wiczer (2023), Blanco, Drenik, Moser and Zaratiegui (2023). By linking

[5]The fact that labor share can be a proxy of operating leverage within firms has been observed in corporate finance and labor studies (Favilukis, Lin and Zhao 2020, Acabbi, Panetti and Sforza 2023). A very similar dynamic is observed in Ai and Bhandari (2021).

human capital accumulation to firm quality we extend the previous analyses to quantify the feedback channels between labor market sorting and fluctuations in aggregate output.[6] In particular, in Carrillo-Tudela and Visschers (2023) and Carrillo-Tudela et al. (2023) workers accumulate occupation-specific human capital and the cyclical behavior of occupational mobility interacts with unemployment duration and workers' careers. In their framework, the collapse of the job ladder following a recession makes the sullying effects of recession long-lasting by dampening the returns to occupational mobility. Similarly, Baley et al. (2022) focuses on cyclical mismatches in tasks. In their setting, workers are uncertain about their ability. Displaced workers might find it optimal to switch occupations throughout recessions, when the cost of switching is comparatively lower. As a consequence, they find recessions to be cleansing. In our analysis we abstract from multi-dimensional skills or heterogeneous occupations. Our focus is on the analysis of long run effects of aggregate risk for output and earnings, career discontinuities and unemployment, allowing workers' productivity to be determined by their job search strategies and the history of their employment relationships.

Jarosch (2023) uses a search model in which workers progressively climb a "slippery" job ladder to gain job security, but accumulate human capital stochastically when employed. In his model the persistent effects derive from the fact that jobs exogenously differ by productivity and layoff risk. Our model manages to match the unemployment risk over the life cycle and by work experience without modeling any exogenous additional characteristic of jobs, but as a by-product of endogenous rational choices on the part of workers.[7] In addition, by modeling directly aggregate risk and a realistic age structure, we characterize output dynamics and hysteresis. Lastly, we show that a model featuring endogenous (match-dependent) human capital accumulation and separations matches considerably better output and unemployment volatilities in the data than a model without those ingredients.

2 Model

2.1 Environment

Time is discrete, runs forever and is indexed by $t \in \mathbb{N}$. We denote future values in recursive expressions by adding a $'$ to them, or index elements by t in non-recursive ones.

[6] For a discussion on the importance of human capital accumulation as a determinant of the dispersion in workers' career earnings see also Bagger, Fontaine, Postel-Vinay and Robin (2014), Lise and Postel-Vinay (2020), Taber and Vejlin (2020), Griffy (2021).

[7] Given the model's treatment of pay dynamics within matches, our research is also related to a strand of literature in labor and finance analyzing the firms' management of liquidity and labor compensation dynamics (Xiaolan 2014; Favilukis et al. 2020; Acabbi and Alati 2022; Acabbi et al. 2023).

The economy is populated by a unit mass of $T \geq 2$ overlapping generations of finitely lived, hand-to-mouth, risk-averse workers and a continuum of risk-neutral entrepreneurs. All agents in the economy share the same discount factor $\beta \in (0,1)$. Each household lives for T periods, with age $\tau \in \mathcal{T} \equiv \{1,2,3,\ldots,T\}$. Workers are either employed, with value function W, or unemployed, with value function U.

Workers maximize lifetime flow-utility from non-durable consumption:

$$\mathbb{E}_{t_0}\left(\sum_{\tau=1}^{T} \beta^{\tau} u(c_{\tau,t_0+\tau})\right),$$

where t_0 characterizes the cohort year of entry into the labor market, and τ characterizes the age of the agent. Consequently, $c_{\tau,t_0+\tau}$ refers to the consumption of workers of age τ in time $t_0+\tau$.

Workers are characterized by heterogeneous human capital levels h, with $h \in \mathcal{H} \equiv [\underline{h}, \overline{h}]$, and are heterogeneous also with respect to their formal education level $\iota \in \mathcal{I} \equiv \{\text{g,s}\}$, which indicates college and high school education, respectively. Both types enter the labor market with a baseline level of human capital drawn from type-specific exogenous continuous distributions. Upon entry in the labor market, $\mathbb{E}[h|\text{g}] > \mathbb{E}[h|\text{s}]$. To account for the different number of education years in the data, graduate workers entry to the labor market is delayed accordingly. Workers exit the labor force when their employment prospects deteriorate below a certain utility threshold, effectively distinguishing long-term unemployment from non-participation.

We model the human capital accumulation process assuming that it depends on the quality of the firm workers are matched with, and on their own initial level of human capital, h. Firms are characterized by different levels of quality $y \in \mathcal{Y} \equiv [\underline{y}, \overline{y}]$. Workers accumulate human capital only while employed and according to a law of motion that is match-specific: $h' = \phi(h,y,\iota,\psi) = g_\iota(h,y) + \psi$, $\phi : \mathcal{H} \times \mathcal{Y} \times \mathcal{I} \times \Psi \to \mathcal{H}$, where g_ι is the deterministic component of the human capital accumulation dynamics, and $\psi \in \Psi$ constitutes a stochastic component. The function g_ι is concave in both its arguments. The deterministic component of human capital accumulation is akin to a "catching-up" of the firm's quality, up to a point when the worker will not be able to learn any more from the match. The only difference across education levels is the speed of the "catching-up", with college-educated workers catching up faster.[8]

Human capital accumulation is risky: at any period any employed worker is subject to the idiosyncratic human capital shock ψ, which enters additively with respect to the deterministic component.[9] The shock affects workers' ability and can amplify, shrink, or

[8]Workers who match with a low-quality firm will see their ability deteriorating with the same g_ι function.

[9]The additive nature of the shock keeps the properties of monotonicity and uniqueness of workers'

even reverse human capital accumulation. We further allow for the possibility that human capital deteriorates while workers are unemployed, according to an arbitrary process g_u.[10]

Firms are modeled as one worker–one job matches, thus abstracting from firm size. Each job match is characterized by a promised utility to the worker $V \in \mathcal{V} \equiv [\underline{v}, \overline{v}]$. We group worker-specific characteristics in a tuple $\chi \in \mathcal{X} \equiv \{\mathcal{H} \times \mathcal{T} \times \mathcal{I}\}$. The aggregate state of the economy Ω is characterized by the productivity level $a \in \mathcal{A} \subseteq \mathbf{R}_0^+$ and by the distribution of agents across states $\mu \in \mathcal{M} : \{W, U\} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{V} \to [0, 1]$. Let $\Omega = (a, \mu) \in \mathcal{A} \times \mathcal{M}$ represent the aggregate state of the economy and let $\mathcal{M}$ represent the set of distributions μ over the states of the economy. Then $\mu' = \Phi(\Omega, a')$ is the law of motion of the distribution. Aggregate productivity is a stationary monotone increasing Markov process, namely $a' \sim F(a'|a) : \mathcal{A} \to \mathcal{A}$, with the Feller property.

2.2 Labor markets

Search is directed. Each labor market is organized as a continuum of submarkets indexed by the expected lifetime utility offered by firms of type y, $v_y \in \mathcal{V}$. The process of starting a firm, which amounts to posting a vacancy at a quality-specific cost $c(y)$, will be described in **Section 2.4**.

The matching function $M(u, \nu)$ for each submarket has constant return to scale and is twice continuously differentiable. The tightness of each submarket in $\mathcal{X} \times \mathcal{V}$ is defined as $\theta = \nu/u$, with $\theta(\cdot) : \mathcal{X} \times \mathcal{V} \times \mathcal{A} \times \mathcal{M} \to \mathbf{R}_0^+$. Job finding rates are defined as $p(\theta(\cdot)) = M(u, \nu)/u$, where $p(\cdot) : \mathbf{R}_0^+ \to [0, 1]$ is a twice continuously differentiable, strictly increasing, and strictly concave function with $p(0) = 0$, $\lim_{\theta \to +\infty} p(\theta) = 1$ and $p'(0) < \infty$. The vacancy-filling probability is defined as $q(\theta(\cdot)) = M(u, \nu)/\nu$, where $q(\cdot) : \mathbf{R}_0^+ \to [0, 1]$ is twice continuously differentiable, strictly decreasing, and strictly convex, with $q(0) = 1$, $\lim_{\theta \to +\infty} q(\theta) = 0$ and $q'(0) < 0$, such that $q(\theta) = p(\theta)/\theta$, and $p(q^{-1}(\cdot))$ is concave.

Upon matching, workers produce according to the twice-continuous increasing and concave production function $f(h, y; a) + x(a) : \mathcal{A} \times \mathcal{H} \times \mathcal{Y} \to \mathbf{R}_0^+$. The $x(a)$ component of the production function is a cost which can depend on the aggregate productivity realization.[11] Workers' compensation is determined by dynamic contracts through which firms deliver a promised lifetime utility, as described in **Section 2.5**.

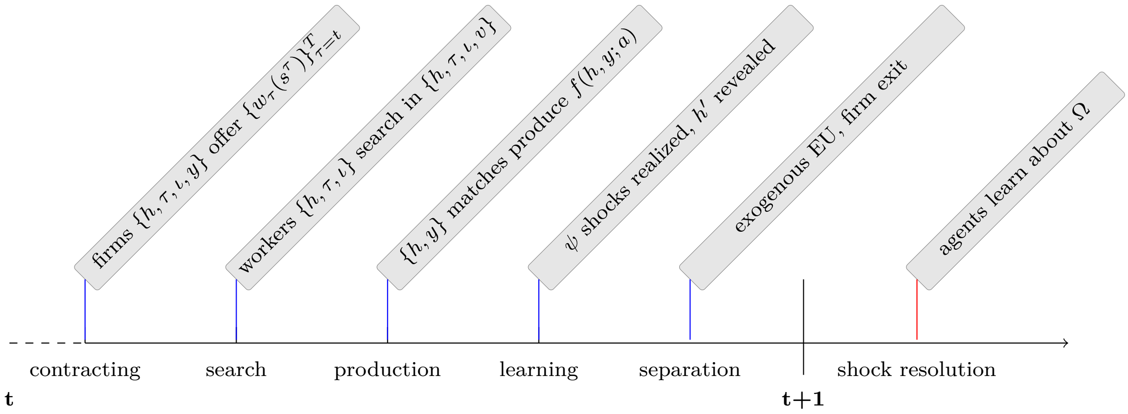
Workers search on the job with probability λ_e. Matches are destroyed each period at an exogenous rate, possibly varying by age, λ_τ. Matches separate also if the worker moves to another firm (poachings), or if the value of the match for the firm becomes negative

search strategies unaltered, which is essential for tractability.

[10] This process might be without loss of generality deterministic or stochastic, and might or might not depend on current human capital h.

[11] This is a reduced-form way of incorporating financial frictions in the model, which make fixed costs loom larger over flow-production in downturns.

Figure 1. Timeline of Worker–Firm Match



(firings). Lastly, unemployed workers whose expected value of re-employment falls below a threshold $\underline{\mathrm{p}}$ are assumed to permanently exit the labor force.[12]

Timing is represented in **Figure 1**. At the beginning of each period an aggregate productivity shock is drawn; entrepreneurs open vacancies across submarkets and post their offers; workers search from unemployment or on-the-job, and move to a new job if the search is successful; production takes place; workers accumulate human capital depending on their employment status and idiosyncratic shock realization; an exogenous share of matches breaks down, while some firms endogenously exit.

2.3 Informational and contractual structure

Firms post fully state-contingent contracts. Each contract prescribes an action for each realization of the history of the worker–firm match. The tuple defining productivity and worker characteristics in a match with any firm y at time t is defined by $s_t = (h_t, \tau_t, \iota, a^t, \mu^t) \in \mathcal{S}^t = \mathcal{X} \times \Omega^t = \mathcal{H} \times \mathcal{T} \times I \times \Omega^t$, that is the worker skill, age, formal education, the history of aggregate productivity shocks, and workers' distributions across their employment history. A given history of realizations between t and k periods ahead is thus $s^{t+k} = (s_t, s_{t+1} \dots, s_{t+k})$. The contract defines a transfer of utility from the risk-neutral firm to the risk-averse worker within the match for all future possible histories of shocks. We define τ_{t_0} as the age at which the worker is hired and T is the retirement age. The history of realizations between t_0, the time of hiring of the worker, and $t_0 + (T - \tau_{t_0})$, the time of maximum duration of the match with the worker before retirement, is thus $s^{t_0+(T-\tau_{t_0})}$.

Histories of workers and productivity shocks are common knowledge, and the future

[12]Workers could also voluntarily decide to quit to unemployment. We experimented with allowing workers quits to unemployment and found that in the model, given plausible parameterizations and in absence of additional shocks, quits to unemployment are never optimal. For this reason and for simplicity we remove the possibility from the model presented in this paper.

realizations of shocks are fully contractible. While the contract is state-contingent, workers' actions are private knowledge in the search stage, so firms are unable to counter outside offers. The contracts offered by firms are then defined as:

$$\mathcal{C}^{\tau_{t_0}} := (\mathbf{w}, \zeta) \text{ with } \mathbf{w} := \{w_t(s^{\tau_t - \tau_{t_0} + t_0})\}_{t=t_0}^{t_0+(T-\tau_{t_0})}, \text{ and } \zeta := \{v_t(s^{\tau_t - \tau_{t_0} + t_0})\}_{t=t_0}^{t_0+(T-\tau_{t_0})}. \tag{1}$$

Firms promise a series of state-contingent wages defined by the series of utility values v_t sought at each node of the history.[13] ζ is the action suggested by the contract, which is bound to be incentive compatible for the worker. The resulting relationship between workers and firms is characterized by a contract with forward-looking constraints. The state space of the worker problem can be expressed in terms of their current lifetime utility, as in Spear and Srivastava (1987), so as to avoid having to keep track of all past histories s^t at each period. The relevant state space is then $\mathcal{X} \times \mathcal{V} \times \Omega$.

2.4 Vacancy creation and free entry

The economy is populated by a continuum of risk-neutral entrepreneurs. Each entrepreneur can invest to reach the desired level of firm quality y. The start-up costs of the firm are priced in terms of the consumption good and they coincide with vacancy posting costs in the frictional labor market. The cost of each vacancy is positively related to the quality of the firm being created through the cost function $c(y)$, which is increasing and strictly convex.[14]

At a generic time t each entrepreneur chooses in which submarket to post the vacancy by offering utility $W \in \mathcal{V}$. Each submarket is characterized by worker characteristics and current utility $(\chi, V) \in \mathcal{X} \times \mathcal{V}$, and we prove later in **Section 2.5** that the firm choice over W uniquely maps upon vacancy posting into firms' qualities $y \in \mathcal{Y}$ conditional on the submarket's characteristics of choice.

We define $J(h, \tau, \iota, W, y; \Omega) \in \mathcal{X} \times \mathcal{V} \times \mathcal{Y} \times \Omega$ as the value function of a firm, which capitalizes all future profits from the match. As entrepreneurs choose the submarkets in which to open a vacancy, they face the following problem:

$$\Pi(h, \tau, \iota, W, y; \Omega) = \sup_{h,\tau,\iota,W} -c(y) + q(\theta(h, \tau, \iota, W; \Omega))[J(h, \tau, \iota, W, y; \Omega)] \tag{2}$$

Given perfect competition, free entry and the possibility for all entrepreneurs to choose

[13] Similarly to Menzio and Shi (2010), Tsuyuhara (2016), and Balke and Lamadon (2022), to guarantee that the problem is well behaved and the firm profit function is concave, the contract will require a two-point lottery, which specifies probabilities over the actions prescribed. We omit it here for conciseness.

[14] We assume that entrepreneurs can borrow from risk-neutral, deep-pocketed financiers to finance the vacancy. As in Herkenhoff (2019) this assumption implies that the cost of credit for entrepreneurs coincides with the risk-free rate.

any possible firm kind y, the expected profits from creating a vacancy are driven down to 0 in submarkets that actually open.This translates into a free entry condition:

$$\Pi(h, \tau, \iota, W, y; \Omega) \leq 0 \text{ for } \forall\{h, \tau, \iota, W, y; \Omega\} \in \{\mathcal{X} \times \mathcal{V} \times \mathcal{Y} \times \Omega\} \tag{3}$$

The equilibrium tightness in each open submarket is:

$$\theta(h, \tau, \iota, W; \Omega) = q^{-1}\left(\frac{c(y)}{J(h, \tau, \iota, W, y; \Omega)}\right). \tag{4}$$

2.5 Firm problem

The firm contract design plays out as a sequential equilibrium game with leader-follower dynamics, in which firms play as the principal/leader and workers are the agents/followers. Workers' limited commitment implies they will search for new jobs whenever they have the possibility to do so. Firms cannot observe poaching offers and thus cannot counter them. The sequence of past histories s^t is common knowledge, and while the firm cannot observe any of actions of its workers, it has enough information to internalize their optimal search policy decisions. Define $\widetilde{p}(\chi, V; \Omega)$ as the optimal retention function and $\widetilde{r}(\chi, V; \Omega)$ as the optimal continuation utility for workers from workers' solution to the on-the-job search problem.[15] The value function of an incumbent firm y in state $(h, \tau, \iota, W; \Omega)$ can be written recursively using the promised utilities as additional state variables as:

$$\begin{aligned} J(h,&\tau, \iota, W, y; \Omega) = \\ &\sup_{\pi_i, \{w_i, W_i'\}} \sum_{i=1,2} \pi_i \Bigg(f(y, h; a) - w_i \\ +&\beta \mathbb{E}_\psi \left[\max\left\{0, \mathbb{E}_\Omega\left[\widetilde{p}(h', \tau+1, \iota, W_i'; \Omega') J(h', \tau+1, \iota, W_i', y; \Omega')\right]\right\}\right]\Bigg) \end{aligned} \tag{5}$$

$$s.t.\ W = \sum_{i=1,2} \pi_i \left(u(w_i) + \beta \mathbb{E}_{\Omega,\psi}\left(\widetilde{r}(h', \tau+1, \iota, W_i'; \Omega')\right)\right), \tag{6}$$

$$\sum_{i=1,2} \pi_i = 1, \tag{7}$$

where **Equation** (6) is the promise keeping constraint ensuring that the current value of the contract is based on the current wage and future utility promises. The firm chooses the wage(s) to be offered in the current period w_i, the utility promises W_i' and the probability π_i in a two-point lottery.

[15]These expressions are going to be made explicit in **Section 2.6**.

Firms commit to the delivery of a utility value to workers, but exit when the present value of future profits becomes negative. Incumbent firms make their exit decisions before the realization of aggregate productivity but after the realization idiosyncratic human capital shocks for the next period.[16] At the beginning of a period both the firms and the workers already know whether the firm will shut down. Exit is therefore completely determined by the current state and can be summarized by a threshold for the aggregate productivity level and can be described as follows.

Definition 2.1 (Exit policy)**.** *The following indicator takes a value of one if the firm does not decide to exit in the following period:*

$$\eta(h, \tau, \iota, W, y; \Omega) = \begin{cases} 1 & \textit{if } a \geq \max\{0, a^*\} \\ 0 & \textit{otherwise} \end{cases}$$

with the productivity threshold defined as

$$a^*(h, \tau, \iota, W, y; \Omega) : \mathbb{E}_\Omega[J(h, \tau, \iota, W, y; \Omega')] = 0. \tag{8}$$

2.6 Worker problem

Given current lifetime utility V, job seekers with characteristics χ have to decide in which submarket to direct their search. Submarkets are indexed by worker type χ and by offered utility v associated to firms' posted vacancies. As discussed in **Section 2.4**, the choice over v will also indirectly determine which firm y the worker matches with, and thus the implied human capital accumulation path. For now, let us assume this (conditional) mapping exists. This amounts to assuming that the function $v(y; \chi, V)$ is a bijective function $f_v : \mathcal{X} \times \mathcal{V} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{M} \to \mathcal{V}$. Upon observing a job offer with utility v, a worker χ with current utility V will be able to infer which firm type y is posting the offer.

A worker of type (χ, V) that enters the search stage has lifetime utility $V + \lambda_i \max\{0, R(\chi, V; \Omega\}$, where the second component of the expression embeds the option value of the search, with R being the search value function, and $\lambda_i = 1$ if the worker is unemployed or $\lambda_i = \lambda_e$ if they are employed. R is defined as:

$$R(\chi, V; \Omega) = \sup_v \Big[p(\theta(\chi, v; \Omega)) \big[v - V \big] \Big]. \tag{9}$$

We denote the solution of the search problem as $v^* = v^*(\chi, V; \Omega)$, and $p^*(\chi, v^*; \Omega) = p(\theta(\chi, v^*; \Omega))$ as the associated optimal job-finding probability. The lifetime utility of an unemployed worker at the beginning of the production stage can be defined as

[16]This amounts to having the firm committing to a state-contingent exit decision in advance of the idiosyncratic shock's realization as in Gomes (2001) and Xiaolan (2014).

$$U(h,\tau,\iota;\Omega) = u(b(h,\tau)) + \beta\mathbb{E}_{\Omega,\psi}\bigg(U(h',\tau+1,\iota;\Omega') + \max\{0, R(h',\tau+1,\iota,U(h',\tau+1,\iota;\Omega');\Omega')\}\bigg), \tag{10}$$

where $b(h,\tau)$ is a skill and age dependent unemployment benefit. Given finite workers' lives, $U(h,\tau,\iota;\Omega) = 0\ \forall(\chi;\Omega) \in \mathcal{X}\times\mathcal{A}\times\mathcal{M}$ whenever $\tau > T$. The corresponding lifetime utility of a worker employed at firm y, with human capital h, age τ, education ι and promised utility V at the beginning of production stage can be expressed as:

$$V(h,\tau,\iota;\Omega) = u(w) + \beta\mathbb{E}_{\Omega,\psi}\bigg(\lambda_\tau U(h',\tau+1,\iota;\Omega') + (1-\lambda_\tau)\Big[V(h',\tau+1,\iota;\Omega') + \lambda_e \max\{0, R(h',\tau+1,\iota,V(h',\tau+1,\iota;\Omega');\Omega')\}\Big]\bigg), \tag{11}$$

where w is the promised wage and $V(h',\tau+1,\iota;\Omega')$ is next period's state-contingent promised utility of remaining in the current firm, which becomes the outside option in the search problem.[17] Again, $V(h,\tau,\iota;\Omega) = 0\ \forall(\chi;\Omega) \in \mathcal{X}\times\mathcal{A}\times\mathcal{M}$ whenever $\tau > T$. Firms internalize incentives embedded in workers' strategies and post wages and utility offers to maximize profits by optimizing retention. This way, future promised utilities incorporate both future wages or option values of search.

The policy functions are uniquely defined and identify target y as long as the bijective mapping between the offered utility v and y given χ exists.[18] The solution of employed workers' on-the-job search problem defines the two equilibrium objects mentioned in **Section 2.5**, which firms internalize in order to incorporate workers' incentive compatibility .

Definition 2.2 (Optimal retention probability and utility return)**.** *The solution to the worker's problem defines a retention function* $\widetilde{p} : \mathcal{X}\times\mathcal{V}\times\Omega \to [(1-\lambda)(1-\lambda_e), 1-\lambda]$ *and a utility return* $\widetilde{r} : \mathcal{X}\times\mathcal{V}\times\Omega \to \mathcal{V}$*:*

$$\widetilde{p}(\chi, V;\Omega) \equiv (1-\lambda_\tau)(1-\lambda_e p^*(\chi, v^*;\Omega)) \tag{12}$$

$$\widetilde{r}(\chi, V;\Omega) \equiv \lambda_\tau U(\chi;\Omega) + (1-\lambda_\tau)\Big[V + \lambda_e \max\{0, R(\chi, V;\Omega)\}\Big] \tag{13}$$

[17]It is here implied that, in case there is an endogenous separation, this future promised value is equivalent to the value of being unemployed.

[18]Proofs of the uniqueness of policy functions and individuals' optimal policy are provided in Appendix Section A.1.

2.7 Equilibrium definition

Recursive Equilibrium. Let $\Theta = \mathcal{A} \times \mathcal{M} \times \mathcal{H} \times \mathcal{T} \times \mathcal{I}$. A recursive equilibrium in this economy consists of a market tightness $\theta : \Theta \times \mathcal{V} \to \mathbb{R}_+$, a search value function $R : \Theta \times \mathcal{V} \to \mathbb{R}$, a search policy function $v^* : \Theta \times \mathcal{V} \to \mathcal{V}$, an unemployment value function $U : \Theta \to \mathbb{R}$, a firm value function, $J : \Theta \times \mathcal{V} \times \mathcal{Y} \to \mathbb{R}$, a series of contract policy functions $\{c_\tau\}_{\tau=1}^T : \mathcal{S}^\tau \times \mathcal{Y} \to \mathcal{C}^\tau$, a bijective mapping between firm qualities and promised utilities at hiring $f_v : \Theta \times \mathcal{V} \times \mathcal{Y} \to \mathcal{V}$, an exit threshold for aggregate productivity $a^* : \Theta \times \mathcal{V} \times \mathcal{Y} \to \mathcal{A}$, a human capital accumulation process $\phi(h, y, \iota, \psi)$, $\mathcal{H} \times \mathcal{Y} \times \mathcal{I} \times \Psi \to \mathcal{H}$, and a law of motion for the aggregate state of the economy $\Phi_{\Omega,a} : \mathcal{A} \times \mathcal{M} \to \mathcal{A} \times \mathcal{M}$ such that:

1. Given the mapping f_v, market tightness satisfies **Equation** (4).
2. The unemployment value function solves **Equation** (10).
3. Search value functions solve the search problem in **Equation** (9) and v^* is the associated policy function.
4. Firm value functions and associated contract policy functions solve **Equation** (5) for each $\tau \leq T$.
5. The exit threshold satisfies **Equation** (8).
6. The law of motion for the aggregate state of the economy respects the search and contract policy functions and the exogenous process of aggregate productivity.

Definition 2.3 (Block Recursive Equilibrium)**.** *A Block Recursive Equilibrium (BRE) is a recursive equilibrium such that the value and policy functions depend on the aggregate state only through aggregate productivity, $a \in \mathcal{A}$ and not through the distribution of agents across states $\mu \in \mathcal{M}$.*

Property 2.1 (Existence of a BRE)**.** *A block recursive equilibrium exists for the model.*

Proof. See **Appendix Section B**. □

3 Discussion

The objective of our model of dynamic sorting is to understand the properties of jobs creation and worker search in a setting with two-sided heterogeneity. The following properties guarantee a high degree of tractability.[19]

[19]We refer the reader to Appendices A.1 and A.2 for a more in-depth discussion of the theoretical properties of the model and the proofs of this section.

Property 3.1 (Unique Bijective Mapping). *Upon matching, firm quality y and utility promises in vacancy postings v are related by a bijective mapping conditional on the aggregate state of the economy, Ω, and workers characteristics (χ, V).*

The previous proposition establishes that workers' directed search toward promised values is equivalent to directed search toward firms' types. We then focus on the properties of the search strategy to get a complete view of how sorting works in equilibrium.

Property 3.2 (Search Monotonicity and Uniqueness). *The optimal search strategy when unemployed, conditional on age τ, formal education ι and the aggregate state Ω, is unique and weakly increasing in workers' human capital h. The optimal search strategy when employed, conditional on age τ, formal education ι and the aggregate state Ω, is unique and weakly increasing in workers' human capital h and current level of lifetime utility V.*

Property 3.2 guarantees that, abstracting from idiosyncratic as well as aggregate shocks, workers sort positively with respect to their human capital. **Property 3.1**, in turn, guarantees that workers with same observable characteristics agree on firms' relative ranking. Firms are thus vertically differentiated, and there is a separating equilibrium whereby workers with different characteristics optimally search in distinct firms.

Because we are interested in how aggregate fluctuations shape the distribution of matches, we now turn to considering how search strategies change depending on aggregate states.

Property 3.3 (Search in Good and Bad Times). *The optimal search strategy is increasing in the aggregate productivity level, a.*

At this point we are able to illustrate one of the main mechanisms of the model, which is represented in **Figure 2**. The figure highlights one way in which aggregate fluctuations modify sorting in the labor market. The value of vacancies posted by each firm in equilibrium changes with the business cycle, as submarkets become less productive and less tight in bad times, also by virtue of the increasing and convex vacancy costs, which do not vary with the cycle. Faced with a lower probability of successfully matching with the firm they would aim to match with in good times, risk-averse workers adjust their search downward.

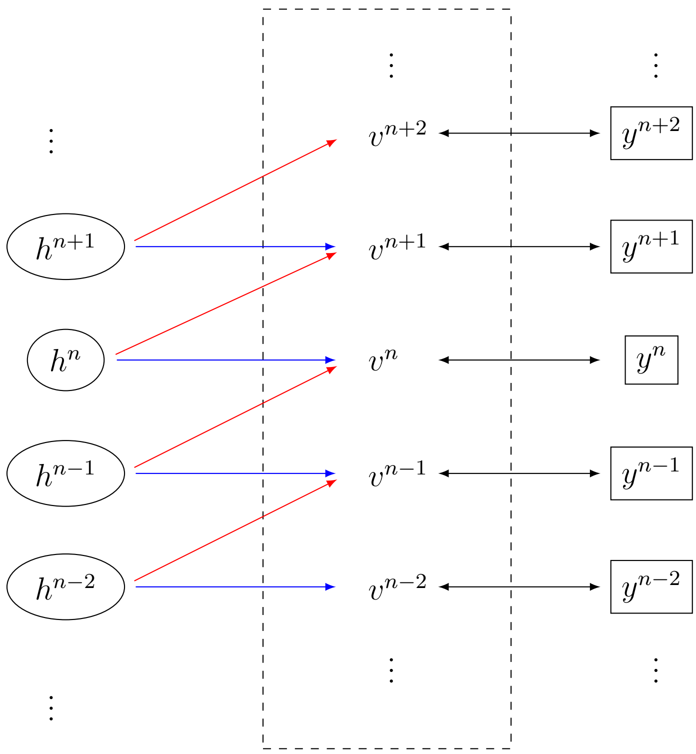
Firms' offers will optimally respond to workers' incentives for on-the-job search.

Property 3.4 (Optimal Retention). *Retention probabilities, $\widetilde{p}(h, \tau, \iota, W; \Omega)$ are:*

(i) increasing in the value of promised utilities, W.

(ii) decreasing in aggregate productivity, a.

Figure 2. Search Behavior.



Note: Schematic representation of labor market sorting along the business cycle. Unemployed workers, ordered by human capital levels, search in bad times and good times toward values offered by the (unique) corresponding firm type, presented as an ordered list with respect to order n. In a recession all workers adjust downwards their target firm quality.

Despite continuation values within the match being procyclical and workers searching more ambitiously in good times, matches will separate more often in expansions as workers transition to new jobs. This is consistent with the data, as employment-to-employment transitions are strongly pro-cyclical. **Property 3.4** highlights another important aspect of the incentives that shape the contract designed by firms: retention grows in continuation values W.

To close the model, we need a rule for surplus sharing between firms and workers, that is, a wage protocol for firms to deliver lifetime utility promises to workers.

Property 3.5 (Wage Protocol)**.** *The optimal contract delivers a wage growth rule that satisfies:*

$$\frac{\partial \log \widetilde{p}(\chi', W_i'; \Omega')}{\partial W_i'} J(\chi', W_i', y; \Omega') = \frac{1}{u'(w_i')} - \frac{1}{u'(w_i)}, \tag{14}$$

with $\chi' \equiv (\phi(h, y, \iota, \psi), \tau + 1, \iota)$ being the definition of individual characteristics and w_i' being the wage paid in the future state, conditional on realizations of idiosyncratic risk ψ and aggregate risk a'.

This result extends the wage equation in Balke and Lamadon (2022) to an environment with two-sided heterogeneity. Wage growth is proportional to the residual continuation value of the match, J and the semi-elasticity of the worker's retention probability to future value promised. Limited liability provides the rationale for inefficient separations. At the same time, it also gives rise to wage rigidity, as it ensures that both elements in Equation 14 are weakly positive if the firm does not close down.[20]

Property 3.6 (Countercyclical Separations)**.** *Conditional on the existing contract and on worker and firm types, there exists an aggregate state a^* below which firms will not continue to operate. The threshold a^* is, all things being equal, increasing in the value promised to workers, and decreasing in worker and firm types.*

A clear implication of **Property 3.6** is that, at the onset of recessions, firms are significantly more likely to lay off workers. In addition, lower-skilled workers and low-productivity firms are more likely to separate in recessions. The counter-cyclicality of separations is a common feature in the data, together with the lower job security enjoyed by workers who are younger, less productive, or provided by firms that are less productive.[21]

3.1 Sorting in equilibrium

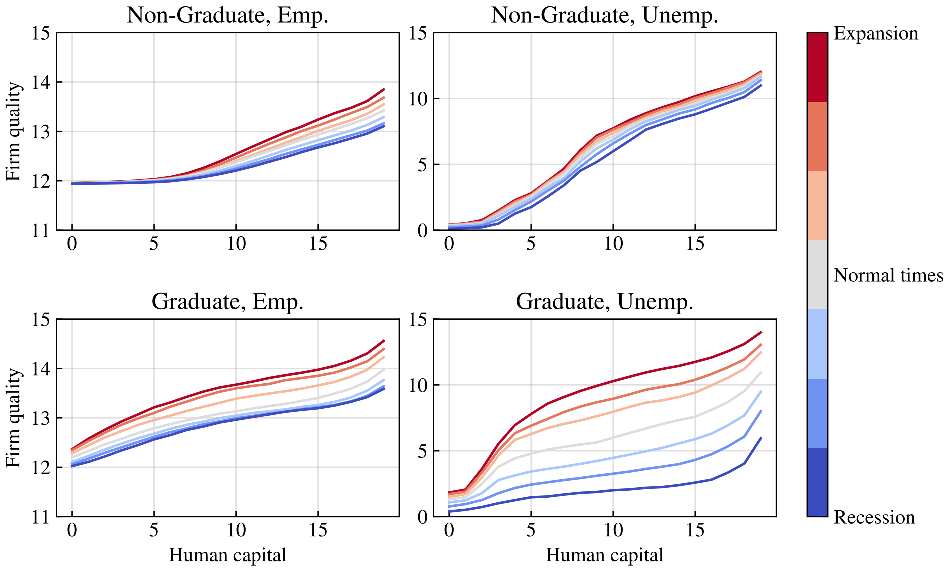
The theory discussed in this section predicts that workers' search is monotonic in individual characteristics and in the aggregate state (see **Proposition 3.2**).

In **Figure 3** we plot the equilibrium mapping between workers' human capital and search behavior resulting from our estimation, for different realizations of the aggregate state. Search is strongly monotonic in both dimensions. Search strategies of college-educated workers are more sensitive to shifts in aggregate conditions. This is due to the fact that expected human capital accumulation is more important for college-educated workers in the model, as it has greater long term effects for their careers. This implies that these workers will have ceteris paribus a greater tendency to search for highly productive matches, with more volatile vacancy creation, in good times. It also implies that they will moderate their search more in bad times, especially if unemployed, in order to minimize their unemployment risk and the consequent human capital deterioration. Because of these dynamics, vacancy creation tends to be of lower quality in recessions, which in turn increases misallocation and leads to a deterioration in sorting throughout the recovery. Our model thus endogenously

[20] Notice that, in the presence of risky human capital accumulation, J will fluctuate together with the human capital levels of the worker even in the absence of aggregate fluctuations. However, because the contract provides insurance to workers, changes in their human capital will have asymmetric effects on wage growth, thus weakly increasing the labor share over time (Ai and Bhandari, 2021).

[21] These facts are observed in the data and modeled in Jarosch (2023). Differently from his setting, these facts emerge endogenously in our framework with directed search, without the need of specifying job security as a contract characteristic.

Figure 3. Search Policies



Note: Search policy function by human capital level and aggregate state, averaged across labor market experience and wage promises.

rationalizes the empirical findings in Haltiwanger et al. (2022), who observe that recessions apparently feature an initial "cleansing" phase followed by a more prolonged "sullying" dynamics as the labor market re-builds its job ladder.

4 Bringing the model to the data

The model features internally and externally calibrated parameters. To estimate the first group of parameters, we target moments from Italian administrative data, provided by the Italian Social Security Administration (INPS), for all years between 1996 and 2018.[22] To obtain model moments, we simulate a population of overlapping generations working for 45 years (180 quarters, from 18 to 63 years old, the legal retirement age for most years in our period of analysis). We then use a simulated method of moments (SMM) approach. This section will first present the quantitative setup of the model, then calibration choices for the parameters that are set externally, and finally our estimation results.

4.1 Calibration and estimation

Quantitative Setup. **Table 1** collects all the functional form choices. We assume a Cobb-Douglas production function and allow for potentially cyclical maintenance costs, captured by the parameter x. We follow Schaal (2017) and Menzio and Shi (2010) in picking a CES function in market tightness. Vacancy creation imposes increasing costs

[22] Details of data sources and construction are discussed in Appendix Section D.

in firm's quality y, according to the convex function $c(y)$. Workers are risk-averse with constant-relative-risk-aversion (CRRA) utility. The human capital production technology is concave in the firm quality, y, which is scaled by a parameter ξ, and in the existing stock of human capital, h. Future human capital is also subject to additive i.i.d. shocks, $\psi \sim \mathcal{N}(0, \sigma_\psi)$. Home production is increasing in the stock of human capital according to the parameter ξ_b. Finally, we allow the exogenous separation rate to be age dependent to capture age-specific aspects of worker quality that are unrelated to business cycles but still empirically relevant. The model is then characterized by seven externally calibrated parameters and by 18 jointly estimated parameters.[23]

Table 1. Functional Forms

Functions	
Production function	$f(y,h) = Ay^\alpha h^{1-\alpha} - x(A-1)$
Job-finding probability	$p(\theta) = \theta(1+\theta^\gamma)^{-\frac{1}{\gamma}}$
Vacancy creation cost	$c(y) = \frac{y^\kappa}{\kappa}$
Utility function	$U(c) = \frac{c^{1-\nu}}{1-\nu}$
Human capital accumulation	$g_\iota(h,y) = (\xi y)^{\phi_\iota} h^{1-\phi_\iota} + \psi$
Home production	$b(h,\tau) = b + \xi_b h$
Exogenous exit rate	$\lambda = \frac{\lambda_b}{\lfloor \tau/4 \rceil}$

Calibration. Preference parameters (discount factor β, and agents' risk aversion ν), and the annualized risk-free rate r_f are set in line with the literature. We calibrate the persistence and volatility of the aggregate shock, (ρ_a, σ_a) by estimating an AR(1) on the detrended series of Italian real total factor productivity (TFP). In addition, workers draw their innate ability and human capital upon entry into the market from an initial distribution. We set this initial distribution of human capital for high school–educated workers as a $Beta(\mu_L, \sigma_L)$. College-educated workers draw their initial human capital from the same distribution plus a constant spread, ϑ. We set the shape and scale of the beta distribution and internally estimate the scaling factor to match the ratio of average initial incomes between the two groups of workers. Finally, to properly account for the empirical age distribution, we weigh simulated data according to the age distribution of the Italian working-age population.[24]

Estimation and Identification. We estimate the remaining 18 parameters via SMM, targeting a set of standard labor market moments: labor market flows by age, as well as their correlations with aggregate output; the profile of wage growth over workers' careers; the average unemployment rate; the average inactivity rate in the Italian labor market; the average degree of sorting between workers and firms; and the distribution of firms' value-added. We define sorting as the average over time of the correlation between

[23] Online Appendix Section E provides more details on the model solution and estimation procedure.
[24] Age weights are constructed following the age distribution of the 2010 census from the website of the Italian National Institute of Statistics (ISTAT).

Table 2. Parameter Values

Parameter	Description	Value
	Externally Calibrated	
ν	Risk aversion	2.000
β	Discounting	0.990
r_f	Real interest rate	0.011
(μ_L, σ_L)	Shape and scale of initial human capital dist.	(2.50,10.00)
(ρ_A, σ_A)	Mean and std of TFP process	(0.95, 0.009)
	Jointly Estimated	
α	Production function elasticity to firm quality	0.556
γ	Matching function	1.092
ϕ	Human capital adjustment rate, High School	0.038
ϕ_g	Human capital adjustment rate, College	0.285
b	Unemployment benefit	1.103
λ_b	Exogenous separation prob., initial	0.119
κ	Vacancy cost	2.440
λ_e	On-the-job-search prob.	0.447
ξ	Scaling factor in human capital accumulation	0.644
ξ_b	UB dependence on human capital	0.063
l	Linear loss of humanc capital while unemployed	0.169
τ_{ee}	Human capital retention after EE	0.866
τ_{eu}	Human capital loss after EU	0.763
x	Cyclical component of cost function	-1.711
$\underline{\text{p}}$	Out of labor force threshold	0.037
σ_ψ	Std of idiosyncratic human capital shock	0.698
ϑ	Initial scaling in human capital distribution	0.351
$\underline{\text{y}}$	Lowest bound of firm distribution	2.831

the firm and worker fixed effects from an AKM model yearly estimated on the Italian administrative data. In the model, sorting is the correlation between firms' and workers' qualities. **Table 2** reports the estimated parameter values.

The model fits employment flows by age, capturing labor market dynamism in the data (see Engbom, 2021). We match the cyclical properties of employment flows to account for the jump in job destruction and the drop in job creation during recessions. The elasticity of the matching function γ is primarily identified by the cyclical moves in employment-to-employment transitions, whereas the counter-cyclical role of separations mostly identifies the component of firms' cost function x. Characteristics of labor market fluidity and dynamism are further disciplined by matching the life cycle profiles of employment flows. They jointly help identifying the parameter that governs the frictional nature of on-the-job search, λ_e, the age-dependent separation probability λ_b, and the (firm-dependent) cost of vacancy opening, κ. The model also reproduces the life-cycle wage growth of Italian workers by education levels. Parameters that govern the earnings dynamics drive the estimates of human capital accumulation parameters, ϕ and ϕ_g, together with the parameters that determine the loss in human capital stemming from employment transitions, τ_{ee} and τ_{eu}.

An issue of modelling two-sided heterogeneity is that we need a way to discipline the relative scale of workers' human capital and firm quality, as well as their cross sectional distribution - which in the model depend on ξ and $\underline{y}$. To do so, we match the ratio between the third and fourth to the first quintiles of the firm value added distribution. Labor market sorting depends on the importance of firm quality in both production and human capital accumulation - thus it primarily identifies the production function parameter, α. The inactivity rate identifies the threshold that determines the exit from the labor force, $\underline{p}$. **Table 3** summarizes the comparison of model and empirical moments.

Table 3. Target Moments

Moments	Mean	
	Data	Model
A. Labor Market Flows		
Employment-to-Employment Transition Rate*	1.3%	0.6%
Employment-to-Unemployment Transition Rate*	4.1%	2.6%
Employment-to-Employment Correlation w/GDP	0.68	0.48
Employment-to-Unemployment Correlation w/GDP	-0.19	-0.17
B. Earnings		
Earnings Growth (College)*	129.7%	149.4%
Earnings Growth (High School)*	60.7%	75.9%
Entry Salary Ratio: College to High School	1.46	1.00
C. Other Statistics		
Unemployment Rate	9.7%	8.4%
Labor Market Sorting	0.40	0.57
Inactivity Rate†	21.9%	23.5%
Firm Value Added Distribution**	1.18	1.38

Note: (*): We match the life-cycle profiles with nine age bins for each profile. The table reports average values. (**): We match the ratio between the third and fourth to the first quintiles. (†) The average inactivity rate is taken from the inactivity rates by age groups from 1996 to 2019 from ISTAT.

4.2 Model and data

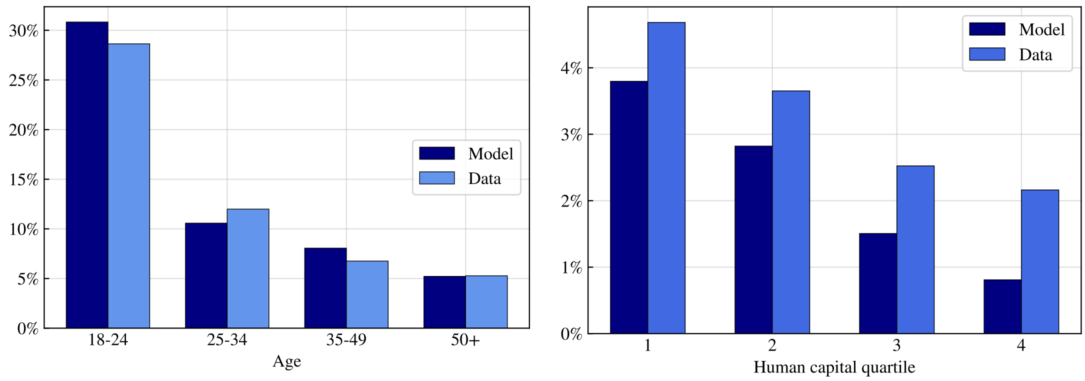
To assess the quality our estimation we check how the model performs in matching a series of untargeted moments both in the cross section and over time as well as both at the macro and micro level.

Cross-sectional properties. While in the estimation we target the average unemployment rate, the model exhibits a very good fit also for the age profile of the unemployment rate (see **Figure 4a**). In addition, **Figure 4b** compares the separation rates by worker types in the model and in the data, highlighting how the model is able to well capture the higher fragility of workers with low human capital, despite not directly targeting these moments in the estimation.

Our baseline model is also able to qualitatively reproduce the distribution of earnings. **Figure 5** displays the cross-sectional distribution of earnings in the data and in the model.

Figure 4. Cross-Sectional Features, Model and Data

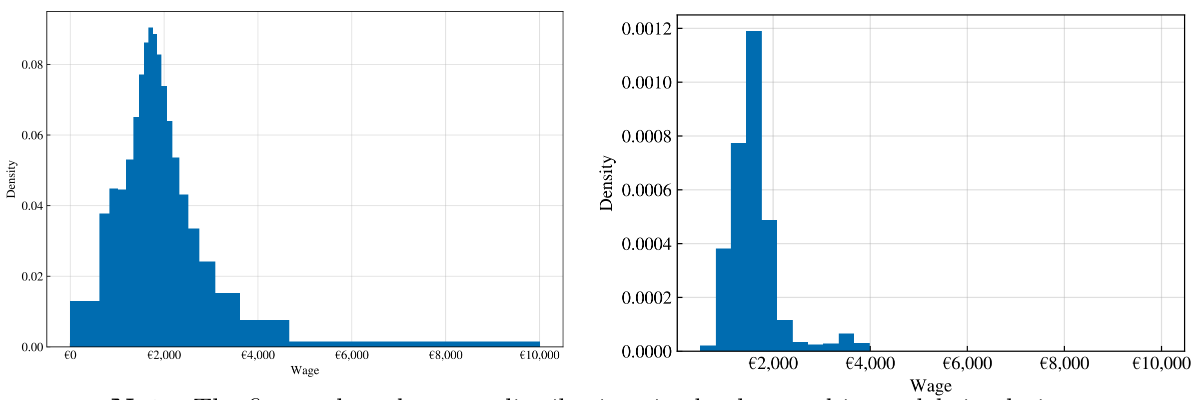
(a) Unemployment rates **(b)** Separation rates by worker qualities



Note: Panel (**a**) plots the unemployment rate by age groups in model simulations and in the data. *Sources:* unemployment rates are taken from the Italian National Statistical Agency (ISTAT). Panel (**b**) reports the average separation rates by worker quality. In the data worker quality is measured by the worker-specific AKM fixed effect. In the model simulations, worker quality is workers' human capital.

Figure 5. Wage distributions

(a) Data **(b)** Model

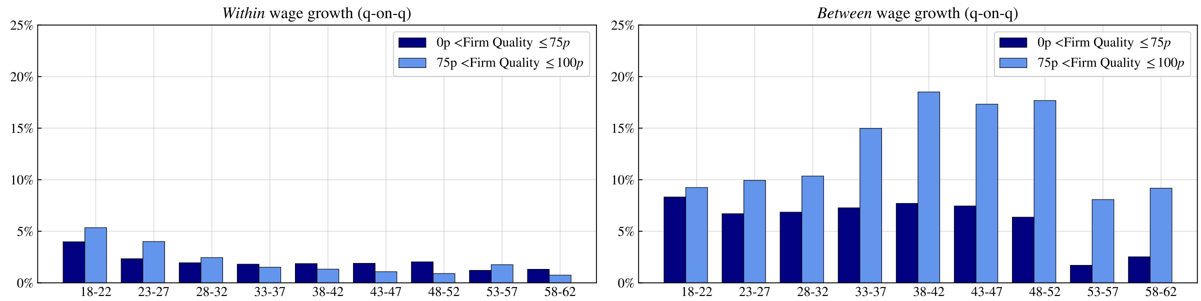


Note: The figure plots the wage distributions in the data and in model simulations.

The empirical wage distribution is centered at slightly below €2,000 and skewed to the left, with most observations below €4,000. What the model fails to generate is the long right tail of wages in the data, which corresponds mainly to managerial figures whose earnings command premia that our mechanism is not meant to capture.

Exploiting the rich features of the model, in **Figure 6** we decompose the average growth in wages *within* jobs, that is within the same job spell, and *between* jobs, that is after a job-to-job transition. Consistent with the data, the model simulation implies that the bulk of wage growth is due to job-to-job transitions. We also observe that average *within*-wage growth is declining in age and firm quality. This is due to the fact that search frictions make it difficult for workers to reach progressively higher-productivity firms. Those firms thus enjoy relatively greater retention probabilities without the need of substantially adjusting wages (as in Gouin-Bonenfant, 2022). The resulting wage profiles for higher-productivity firms are flatter. Given that the highest skill workers tend to be older, one obtains decreasing wage growth over the life cycle and with firm quality.

Figure 6. Within vs. Between Wage Growth by Age and Firm Quality in the Model



Note: The figure plots the average wage growth, by age and firm quality, *within* employment spells and after employment-to-employment transitions (*between*). For the *betweeen* component, the firm quality quartiles are computed on the distribution of origin firms.

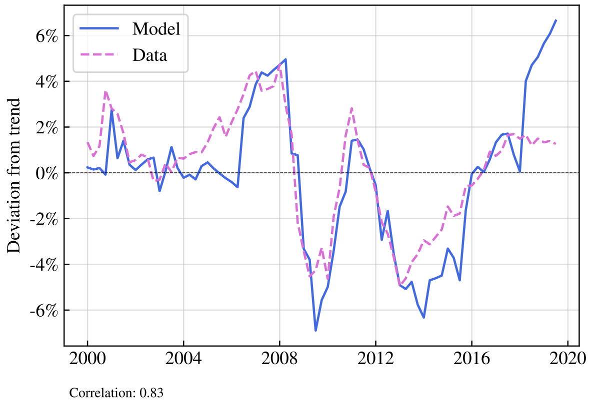
Table 4. Wage Growth of Movers vs. Stayers: Data and Model

		Age				
		18-27	27-36	36-45	45-54	54-62
Data	Between	7.1%	4.8%	3.7%	2.9%	2.2%
	Within	1.7%	1.2%	0.9%	0.6%	0.3%
Model	Between	7.2%	7.4%	10.6%	8%	2.0%
	Within	2.5%	1.0%	0.8%	0.7%	0.3%

Note: The table reports the median yearly wage growth for movers (between wage growth) and stayers (within wage growth). Data: INPS, for all years between 1996 and 2018

The importance of sorting for human capital accumulation and workers' careers can be validated by measuring the correlation of workers ex-post wages with their ex-ante employer quality after an employment to unemployment to employment (EUE)

Figure 7. GDP: Model and Data



Note: The figure plots the cyclical components of real GDP for Italy and for a model simulation in which the TFP process is matched to the Italian TFP realizations from 2000 to 2019, both series are detrended using an Hamilton filter (4 lags and 8 leads) and their correlation is robust to the choice of the filter.

transitions.[25] The correlation is increasing in previous firm qualities, indicating that workers benefit from employment in good firms even once the match is dissolved.[26]

Time series properties. Replicating aggregate time-series properties of the data provides an additional validation of the channels in the model. We use the detrended quarterly series of Italian TFP, and project it on a discrete grid to simulate a series of aggregate shocks in the model. The model tracks the empirical series of GDP quite well, capturing the peaks and troughs as well as the overall behavior of the empirical series. Notably, together with matching well the volatility of output (the standard deviation of detrended log-output is approximately 3% both in the model and in the data) the model is able to generate also a realistic volatility for the unemployment rate - 1.6% in the model versus 1.4% in the data.[27]

The model is also able to replicate the long-run effects of business cycles on workers' career outcomes at the micro level. In particular, we adapt the reduced-form models proposed in the literature on the effects of recessions on labor market entrants (Kahn 2010, Schwandt and von Wachter 2019) and we run it on both the Italian administrative data and on a model-simulated panel.[28] Consistently with the literature, entering the labor market in a downturn is associated with persistent losses in earnings. As shown in **Figure 8**, our baseline model is able to generate scarring effects that, on average, are

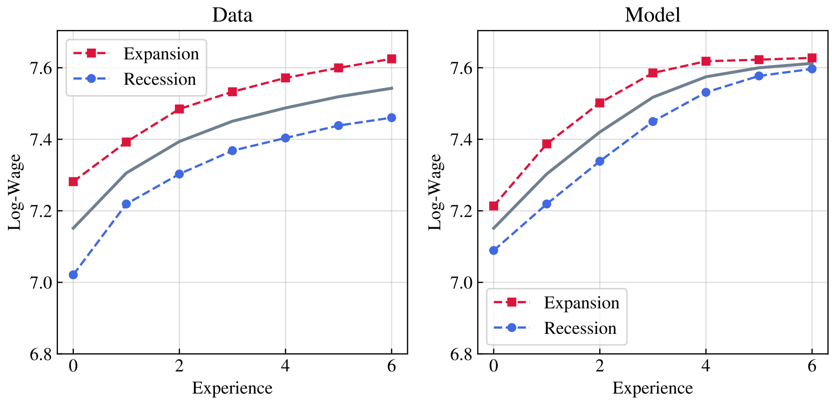
[25] This is an adaptation of Herkenhoff et al. (2018)'s analysis to our model setting. In their paper, they rely on co-workers wages a proxy for firm quality. Given the nature of our framework, we use value-added per employee in the data to measure firm quality, and we control for workers' pre-transition wage.

[26] We report the estimated correlations in the **Table F.3**.

[27] To compute the volatility of unemployment we replicate the exercise in Shimer (2005). Specifically, we remove a slow moving trend from the time-series of unemployment, by filtering the quarterly series with an Hodrick-Prescott filter with smoothing equal to 10^5.

[28] In these empirical specifications we control for age, period, and cohort effects, proxied by the cyclical realization of GDP at cohort entry. We report the empirical estimates in Appendix Section F.

Figure 8. Scarring Effect of Recessions



Note: The figure plots the wage profiles estimated on the data and on model simulations for cohorts of workers entering the labor market. The counterfactual profiles for expansions (recessions) are obtained considering a positive (negative) two standard deviation realization of cyclical GDP.

approximately 25-30% of those observed in the data. The model matches reasonably well both the magnitude and the dynamics of the scarring effects of business cycles.

4.3 Alternative specifications

Our theoretical framework features two specific modeling choices, which depart from previous models in the literature. Firstly, we characterize endogenous separations when a match is no longer profitable. Secondly, and more importantly, we depart from the standard human capital production function assumptions, by making it dependent on firm quality. To assess the significance of these assumptions, we compare our baseline model with two alternative specifications.

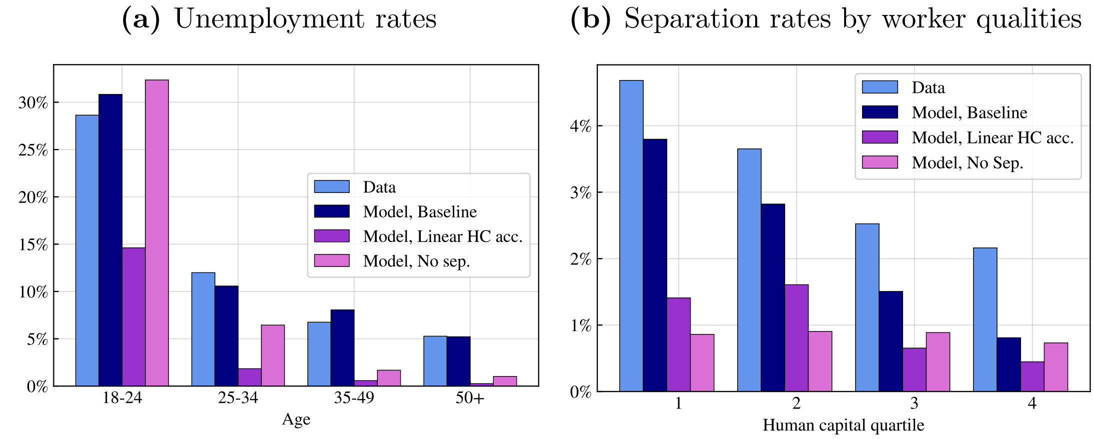
The first alternative, which we refer to as the "linear human capital accumulation" specification, assumes that human capital accumulation is independent of firm quality, and takes place as long as the worker remains employed. In this version, each worker type accumulates human capital according to a random walk with a linear trend.[29]

In the second alternative specification we eliminate the possibility of endogenous separations initiated by firms, by having firms continuing to operate even with negative values and the same wage policy. In this "no (endogenous) separations" specification, matches between firms and workers continue even if the expected value of remaining in the match turns negative, effectively eliminating endogenous separations in firm-worker relationships. However, exogenous separations can still occur, governed by the parameter λ_b.

For both alternative specifications, we follow the same procedure outlined in the

[29]This is a common way of introducing human capital accumulation in search models (Herkenhoff et al., 2023, Jarosch, 2023).

Figure 9. Cross-Sectional Features, Alternative Specifications



Note: Panel (**a**) plots the unemployment rate by age groups for the data and for alternative models: one in which the human capital accumulation is independent from firm quality, and one where there are no endogenous separations initiated by firms. Panel (**b**) reports the average separation rates by worker quality for model and data. Data is as in **Figure 4**.

previous paragraphs and re-estimate the models using the same set of moments presented in **Table 3**. By comparing the results of these alternative specifications to our baseline model, we can gain insights into the importance of the assumptions regarding separations and firm-dependent human capital accumulation.

Panels 9a and **9b** report the unemployment rate by age and the separation patterns for the alternative models. It is clear from the figure that, although the models are re-estimated to achieve the best possible fit on the same moments used in the estimation of the our baseline model, detaching human capital accumulation from sorting to firms makes the model unable to capture the important cross-sectional patterns for both unemployment and separations.

Time Series Properties and the Shimer Puzzle. The correlations between the simulated GDP and unemployment series in the baseline model and the two alternative models are presented in **Table 5**. In the case where human capital accumulation is independent of firm quality, the model fails to replicate the co-movements of unemployment and the cyclical component of GDP observed in Italian data. On the other hand, when we eliminate endogenous separations initiated by firms, the model is still able to reproduce output dynamics and the correlation between unemployment and cyclical GDP. However, the model is unable to reproduce unemployment dynamics from the data.

The table shows that both our key modeling ingredients are necessary to achieve a good fit with the data. The baseline model exhibits a unique capability to capture the volatilities of output and unemployment. This is a feature that search and matching models typically struggle to generate (see Shimer 2005). By comparing the baseline model with alternative configurations, we delve deeper into the underlying causes of this

Table 5. Co-movements at business cycle frequency

(a) Correlations between model generated and data time series

	Model		
	Baseline	Linear Human Capital Acc.	No Separations
Aggregate output	0.83	0.86	0.83
Unemployment	0.40	-0.23	0.07

(b) Correlation with aggregate output

	Data	Model		
		Baseline	Linear Human Capital A..	No Separations
Unemployment	-0.75	-0.89	0.30	-0.62

(c) Output and unemployment volatility

	Data	Model		
		Baseline	Linear Human Capital Acc.	No Separations
Aggregate output	2.3%	2.9%	1.7%	2.1%
Unemployment	1.4%	1.6%	0.2%	0.3%

Note: The table reports the correlation between model simulated series and their data counterparts (Panel 5a), the correlations with aggregate output in the model and in the data (Panel 5b) and the standard deviations of output and unemployment (Panel 5c). The simulations are obtained feeding the model with a series of TFP that matches the realizations of Italian TFP from 2000Q1 to 2019Q4. All series have been detrended. Panel 5a and 5b: Hamilton filter (4 lags and 8 leads); Panel 5c: Hodrick-Prescott filter (smoothing equal to 10^5, as in Shimer (2005)). We report the correlations for our baseline model and for two alternative ones. One in which human capital accumulation is linear (i.e. $h_{\iota,t+1} = \phi_\iota + h_{\iota,t} + \epsilon_{\iota,t}$) and one in which we do not allow for endogenous separations. Both alternative models have been re-estimated on the same moments used for the baseline estimation.

emergent feature.[30]

When human capital accumulation does not depend on firm quality, the inherent value of being employed *per se* grows compared to the value of aiming at more productive matches. Search strategies adapt accordingly, shortening the average spell of unemployment. Workers accumulate human capital with tenure regardless of firm quality, so the match value rarely becomes negative during recessions even for low quality firms. Separations between workers and firms become less frequent in this setup, and occur only for older low-skill workers who have limited potential for further growth. As a result, matching the unemployment level in the model becomes harder.

As the model struggles to characterize endogenous separations along the life cycle, the estimation procedure leads to trying to match unemployment levels by means of a higher likelihood of exogenous separations (λ_b). These exogenous separations are acyclical, so

[30] Schaal (2017) provides a notable exception where a model with idiosyncratic, time-varying volatility can also generate empirically consistent volatilities of output and unemployment.

the model loses its ability to match at the same time unemployment volatility and level. This reasoning underscores why a human capital production function independent of firm quality would miss critical labor flows dynamics within our framework. It also emphasizes, more in general, the importance of accounting for heterogeneous patterns of human capital accumulation to adequately capture the volatility of unemployment.

For very similar reasons, a model incorporating our baseline human capital accumulation dynamics but not allowing firm-initiated endogenous separations also fails in matching unemployment volatility. A model in which there is no cyclical increase in separations due to firm closures in recessions, and for which the probability of losing a job is by construction almost exactly equal across skill and age groups, cannot match unemployment volatility while contemporaneously matching its average level.

5 Anatomy of recessions

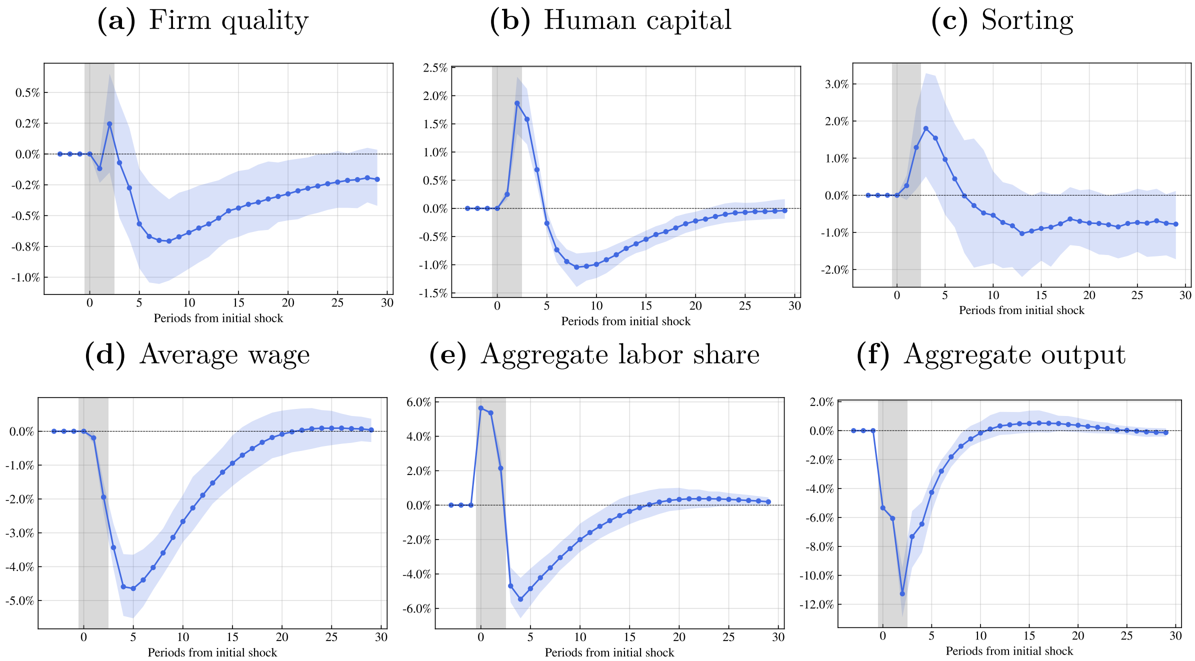
The model developed in **Section 2** can be used to analyze 1) how aggregate shocks propagate through the economy through changes in firm and worker sorting, and 2) what channels are responsible for their persistent effects on aggregate output.

To illustrate these points we provide a series of decompositions of the economy's cumulative impulse responses to a TFP shock. We do this by first computing generalized impulse response functions. We compare a series of simulations of the model with aggregate shocks to otherwise identical counterparts, for which the only difference is that, at some exogenous time chosen to be the same for all simulations, the economy experiences three consecutive negative realizations of the TFP process.[31] We then focus on labor market outcomes of affected cohorts and the response of aggregate GDP throughout these specific recessions. We illustrate how the shock propagates in **Figure 10** and we then provide a decomposition of the response of aggregate output into sorting, human capital and displacement channels in **Figure 11**.

Shock propagation. In **Figure 10a** and **Figure 10b** we show the generalized impulse response functions of average firm quality and human capital. The onset of the recession is characterized by a sort of "Schumpeterian" dynamics, as implied by the initial marginal increase in the levels human capital and firm quality. However, the recession has persistent negative effects on these measures, with the average quality of firms active in the economy remaining approximately 0.5% below the no-recession economy even five years after the end of the recession. The average human capital in the long run recovers its pre-recession level, despite the initial increase and a persistent decrease through the recovery period. **Figure 10c** shows our measure of sorting, the

[31] The cumulative drop in TFP is approximately 15% (equally split in each quarter) .

Figure 10. Recession Experiment



Note: The panels in the figure plot the ratios of the aggregate variables between an economy in which we impose a three-quarter negative TFP realization and an economy without aggregate shocks, that serves as a benchmark. The gray shaded area are the quarters in which TFP is below trend, while the blue shaded area are the 90-10 quantile ranges across one hundred model simulations.

correlation between firms and workers' quality, relative to the baseline simulation. After a short-lived improvement, firm-worker sorting is persistently dampened by the recession, remaining below its pre-recession levels even five years after the onset of the recession. The prolonged reduction in the quality of the factors of production increases the persistence of the initial shock on output beyond the original duration of the recession. In fact, **Figure 10f** shows that two years from the start of the recession, aggregate output its still approximately 1.5% below its counterfactual level.

The recession has a strong effect on both the average wage and the average labor share in the economy, as shown in **Figure 10d** and **10e** respectively. Due to the downward wage rigidty embedded in the contracts, average wages drop less than the average firm output. This allows for a brief spike in the labor share that quickly declines and remains below the counterfactual economy for a long time. The decline occurs for two reasons. First, the matches that form in the recovery period are subject to the sullying in firm and human capital qualities, lowering the average levels of starting wages. Second, due to the back-loading of compensation embedded in contracts, new contracts feature initially lower labor shares and greater profit shares for firms.

Another way to look at the sullying effect of recessions is to look at the different quality of jobs created along the business cycle. **Table 6** performs an exercise similar to Moscarini and Postel-Vinay (2016): we calculate the average tenure and wages for

Table 6. Job characteristics by aggregate state at their inception

	Recession	Low-TFP	Normal	High-TFP	Boom
Tenure	1.00	1.01	1.05	1.14	1.10
Wage	1.00	1.04	1.10	1.17	1.24

Note: The table reports the average duration, wage for jobs that begin in different aggregate states. Recession values are normalized to one.

jobs created at different levels of the aggregate shock. We can see that jobs created in recessions pay on average 10% less than in "normal" times, and 24% less than in "boom" times. Job stability is also affected, with jobs created in recessions lasting 5% to 14% less than when the economy is running at full potential.[32]

State-dependence. We can exploit the variation across the simulations used in **Figure 10** to check if specific states of the economy are correlated with the severity of the recessions, both in terms of cumulative losses and their persistence. We do this by running a series of regressions in which we include the main moments of the *pre-recession* distributions of four important variables of the model: firm quality, human capital, the average wage and labor share, plus a measure of labor market sorting. For each distribution, we consider the first four moments and examine their correlation with three model outcomes: i) short-term losses, which represent the cumulative output response one year after the shock; ii) medium-term losses, which capture the cumulative output response three years after the shock; and iii) the persistence of recessions, measured by the number of quarters from the onset of the shock until the output impulse response function (IRF) returns to zero. In total, we include all seventeen states in the regression analysis, and for ease of presentation, we present the results for the largest coefficients in **Table 7**.[33]

Output losses resulting from negative productivity shocks are primarily influenced by the first moments of the distributions. Specifically, if a recession hits an economy with firm quality one standard deviation higher than the average, the short-term output loss would be significantly stronger, reaching nearly seven standard deviations, and the medium-term loss would be around six standard deviations. This implies that when average firm quality is 1pp higher, the economy experiences larger cumulative output losses of approximately 1.5 and 2.1 percentage points, respectively. In contrast, a 1pp higher level of human capital mitigates the output loss, resulting in a smaller decline of 2.1 percentage points in the short term. However, this higher level of human capital extends the recovery period by approximately 2.6 quarters.

An average labor share 1pp higher instead, would imply a increase of output losses

[32] *Normal times* are periods in which TFP is at its long-run average while *Booms* are periods in which TFP is three standard deviations above average.

[33] We report the full regression tables in Appendix F, Table F.2.

Table 7. State Dependence: Largest Coefficients

(a) Mean

	Output Response		Persistence
	Short-term	Medium-term	
Firm Quality	**-1.6*****	**-2.1****	-0.5
	(0.6)	(1.0)	(1.0)
Human Capital	**2.1****	1.6	**2.6***
	(0.8)	(1.4)	(1.4)
Wage	0.1	0.3	-0.0
	(0.4)	(0.6)	(0.6)
Labor Share	**-2.4*****	**-4.1*****	**1.8***
	(0.6)	(1.1)	(1.1)
Sorting	**2.1****	**3.4****	0.9
	(1.0)	(1.7)	(1.6)
R^2	0.6	0.5	0.1
N	100	100	100

(b) Skewness

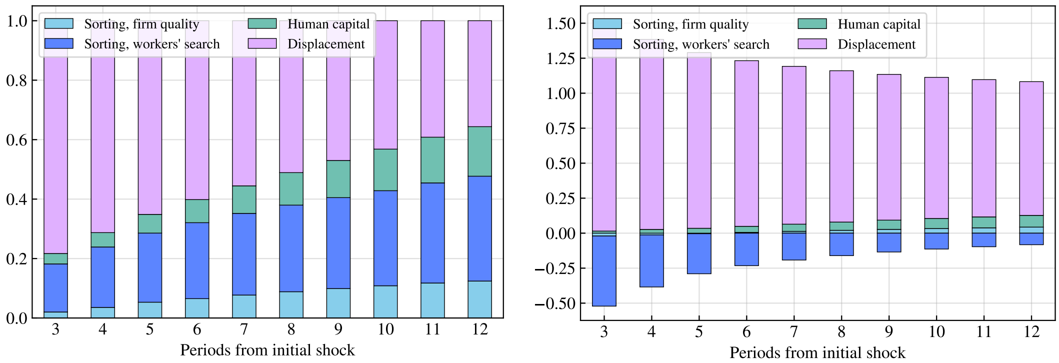
	Output Response		Persistence
	Short-term	Medium-term	
Firm Quality	**-0.6****	**-0.9***	-0.2
	(0.3)	(0.5)	(0.5)
Human Capital	1.2	0.3	**2.6***
	(0.9)	(1.6)	(1.5)
Wage	**-0.4****	-0.3	**-0.7****
	(0.2)	(0.3)	(0.3)
Labor Share	0.3	0.3	**0.9***
	(0.3)	(0.5)	(0.5)
R^2	0.6	0.5	0.1
N	100	100	100

Note: Standard-errors in parenthesis. p-value: * < 0.1, ** < 0.05, *** < 0.01. The table reports the largest coefficients of regressing model outcomes after the shock on moments of relevant endogenous variables before the shock (one year average). The results are scaled to report the effect of changing the regressors by 1pp. Sorting is computed every quarter as the correlation between firm and worker quality and reported in the same table of the means even if technically is not a moment of a distribution. Model outcomes are: i) short-term cumulative output response (1 year after the shock); ii) the medium-term cumulative output response (3 years after the shock); and iii) the persistence of the shock (number of quarters before the output IRF is back at zero). For ease of exposition, the results are grouped by moments but the coefficients are computed including all moments in the same regression, **Table F.2** reports all the coefficients. The simulations are the same underlying the exercise in **Figure 10**.

of 2.4pp in the short-term and of 4.1pp in the medium-term as well as an increase in the length of the recovery by almost two quarters. It is important to highlight the statistical significance of this result. In fact, despite already accounting for the moments of the wage distribution, the labor share plays a crucial role in determining the severity of losses following the TFP shock. This is due to the fact that the labor share incorporates information about two crucial features for our model economy: the tenure distribution and the resulting quality of active job matches. In our contract formulation, highly tenured workers command higher wages and are situated in the flatter part of their human capital accumulation profiles within the match. Consequently, these workers are more vulnerable to separations, which contributes to the labor share's influence on output losses. This relationship is also reflected in the significant coefficient of skewness for the labor share. An economy with labor share skewness one standard deviation higher than the average experiences a slower recovery, taking approximately one additional quarter (0.9 quarters) to reach before recession output levels. On the contrary, a 1pp higher skewness in the wage distribution would reduce the length of the recovery by 0.7 quarters. This difference highlights the importance of taking into account the distribution of match qualities and the underlying tenure structure in the economy. In our model, the labor share is a sufficient statistics for these important dimensions. This result connects our model to previous findings in the literature (Engbom, 2021, Ai and Bhandari, 2021), suggesting that more fluid and dynamic labor

Figure 11. Decomposition of cumulative impulse response functions

(a) Aggregate output **(b)** Average wage



Note: The figure shows the relative importance of each transmission channel compared to the baseline recession for the cumulative response of GDP and the average wage in the three years after the onset of the recession.

markets, characterized by lower tenures, might prove more resilient to recessions on an aggregate level.

5.1 Decomposing recessions

What explains the amplification and persistence of recessionary shocks? Different competing channels are at play. The first, which we call the human capital channel, captures the human capital accumulation that does not take place because of the recessionary event. The second, the sorting channel, amounts to the different joint worker-firm distributions that emerge in the periods following the shock, both because of different search strategies and different match formation along the cycle. Finally, the standard displacement channel captures the job destruction that takes place because of the negative shock and its spillovers. We decompose the effects of recessions in the model economy by shutting down each channel at a time and then comparing the resulting dynamics to the one of a baseline recession.[34]

Figure 11a decomposes output dynamics after a negative shock as driven by our four channels. The displacement channel is the main driver of the recessionary dynamics on impact, explaining the majority of output losses on impact and in the first year after the shock. During the recovery, however, the loss in output can be explained via a combination of a lower search quality from workers as well as a relatively lower firm quality for those that re-match during the transition. Recovery from displacement, while not immediate

[34]Specifically, for the search component of the sorting channel we run model simulations in which the post-recession job-finding probabilities are those associated with search in the baseline simulation. Similarly, for the firm quality component of the sorting channel we keep employed workers in the same firm quality they have in their baseline simulation. For the human capital channel we erase human capital losses by forcing workers' human capital in our counterfactual simulation to be the same as in the baseline one. The displacement channel is obtained residually.

as search is frictional, is relatively fast, in part because unemployed workers have lower reservation wages.

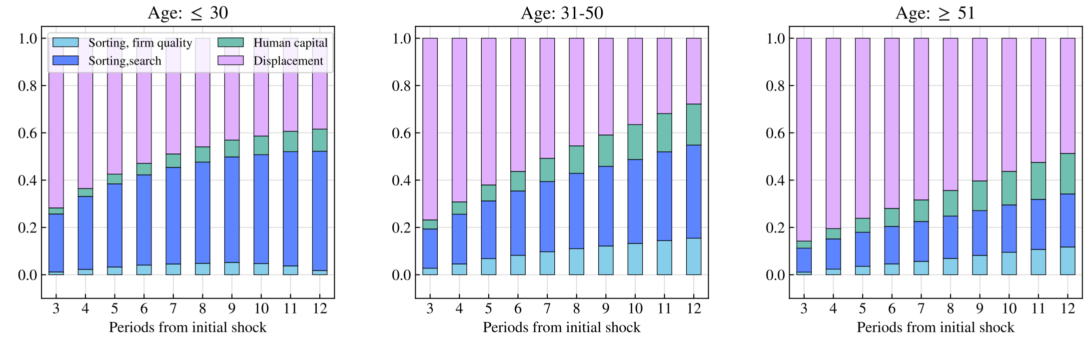
We find that in the medium run, the sorting and human capital channels become more important, and contribute to the persistence of recessionary events. Importantly, the fact that workers search towards lower quality firms in the aftermath of the recession is crucial in explaining the drop in output. Worsened workers' search alone accounts for approximately 20% of output losses in the first year of the recovery and up to 35% three years after the start of the recession. While negligible in the short-run, the importance of lower firm quality builds up over time, amounting to approximately 12% after three years from the onset of the negative TFP shock. Human capital losses, instead, account for approximately 5% in the short-run and up to 17% at the three-years horizon. Taken together, the sorting and human capital channels explain more than 60% of cumulative output losses three years after the shock.

Contrary to the decomposition for aggregate output, the dynamics of average wage are dominated by the displacement effect, as shown in **Figure 11b**. The large drop in wages following the onset of the recession is linked to the fact that workers move from employment to unemployment. The human capital channel gains importance over time but remains of second order importance. Interestingly, the effect of the sorting channel linked to workers' search is strongly negative at the beginning of the recovery (i.e. shutting down this channel would exacerbate the drop in average wage). This reflects the fact that when workers search facing the same job-finding probability of the baseline simulation a larger fraction of workers with relatively low human capital are pulled back into the labor market sooner than in the baseline, decreasing the average wage.

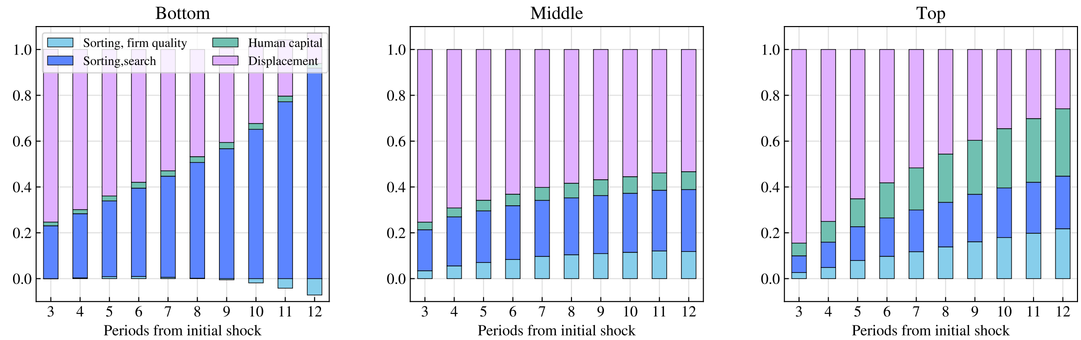
Interestingly, the three channels play different roles in different parts of the workers' human capital or age cross-sectional distributions (see **Figure 12**). The displacement channel matters the most for the left tails of both distributions, as older and low-skilled workers are more likely to be separated. The sorting and human capital channels are instead more relevant for workers that are either young or in the right tail of the human capital distribution. For these workers, as a matter of fact, allocative efficiency and long-run human capital accumulation are particularly important. This intuition is confirmed also by the progressive reduction in importance of the search component of the sorting channel as workers' age increases. This has important implications for the fragility of the economy: as the human capital distribution shifts to the right, recessions become less severe on impact, but might become more persistent and have greater long-run effects thought human capital scarring and firm quality match dynamics. As shown in **Table 6**, jobs that begin in expansions guarantee both higher wages and higher tenure and human capital accumulation within the spell.

Figure 12. Decomposing aggregate output's cumulative response across the age and human capital distributions

(a) Age groups



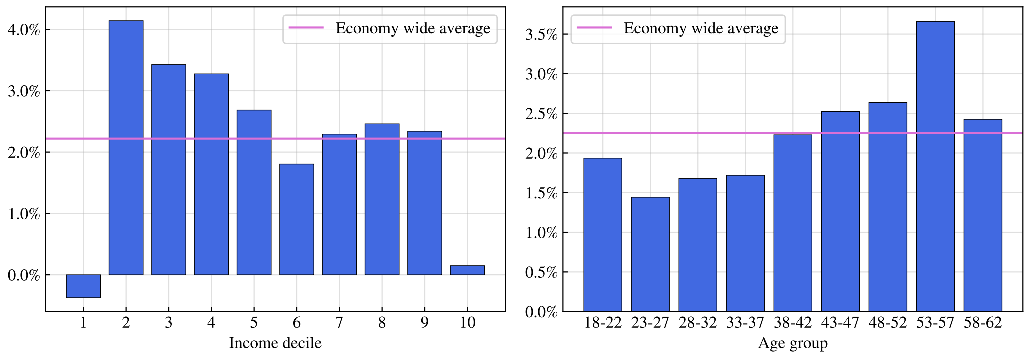
(b) Human capital terciles



Note: For each age group, Panel (**a**) shows the relative importance of each transmission channel compared to the baseline recession for the cumulative impulse response of GDP across different age groups in the two years after the onset of the shock. Panel (**b**) plots the same decomposition across human capital terciles.

Figure 13. Business Cycle Costs

(**a**) By income deciles (**b**) By age groups



Note: Panel (**a**) and (**b**) plot the reduction in consumption-equivalent utility due to aggregate fluctuations by income deciles and age groups.

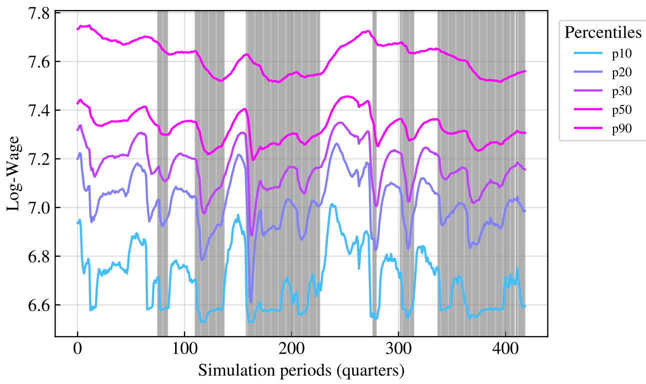
5.2 The costs of recessions

Costs of Business Cycles. Influential work by Lucas (1987) argues that welfare gains from reducing business cycle volatility are negligible, and quantifies them in less than 0.05 percentage points of aggregate consumption. By doing an analogous calculation with our model, we estimate the cost of business cycles to be, on average, greater than 2 percentage points, almost two orders of magnitude above the Lucas estimate. Our estimate is quantitatively close to the one in Barlevy (2004), who first theorized the potentially sullying effects of recessions.

The richness in heterogeneity in our model allows us to estimate how the welfare costs of business cycles vary along the income distribution. We can thus show how welfare costs of business cycles crucially interact with income inequality. **Figure 13a** plots the cost of business cycles for different income deciles. What emerges is that the extreme deciles, the first and the tenth, bear very little to no costs from business cycle fluctuations. For different reasons, both deciles are less affected by separation risk due to aggregate fluctuations: while the bottom decile mostly comprises very low skill unemployed workers, the top decile includes extremely high skill workers, who bear very little risk of losing their job. The welfare costs of business cycles are, however, much larger for all intermediate deciles, and have an interesting U-shape. Up until the seventh decile, the costs of aggregate fluctuations decrease with income: lower incomes are associated with more-fragile jobs and to a higher likelihood of unemployment in recessions. For the last three deciles, costs of business cycles are instead associated with long-run impacts on careers. For workers closer to the top of the income distribution, recessions can have strong negative effects, as the deterioration in sorting and the ensuing lower human capital accumulation prevent them from getting to their best possible employment.

Inequality dynamics. Heathcote et al. (2020) show that recessions have a persistent

Figure 14. Inequality dynamics



Note:The figure reports the dynamics of income for different percentiles of the income distribution.

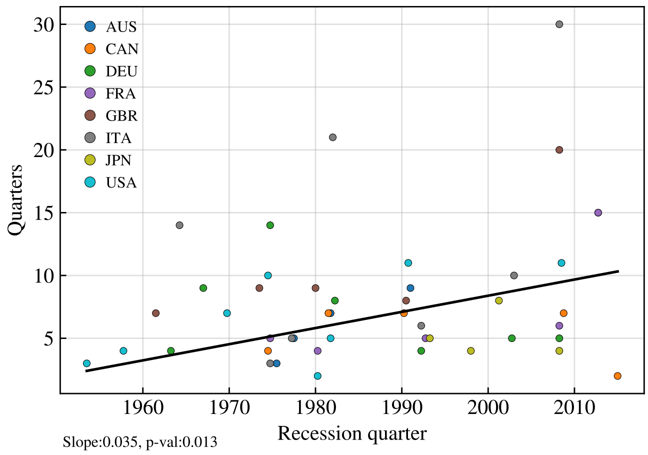
effect on inequality, affecting the earnings of workers in the left tail of the income distribution the most. **Figure 14** illustrates the dynamics of inequality around business cycles by displaying the pattern of losses across the earnings distribution. Recessions hit the poorest workers the hardest, and worsening job prospects push some of them out of the labor force.

A prediction of the model that departs from existing literature is that the persistence of earnings losses varies across the distribution: while workers with less human capital display more volatility in earnings (mostly due to displacement, see **Figure 11b**), the impact on workers with high human capital is dampened but quite persistent. The presence of limited commitment to job matches by firms creates a subset of workers who regularly undergo shorter spells of employment. Their layoffs exhibit strong counter-cyclical behavior (a pattern highlighted in Jarosch 2023). Due to match-specific human capital accumulation, the opportunity for workers to recover from low levels of human capital depends on climbing the job ladder. However, falling off the lowest rungs of the job ladder is more likely than from the highest rungs. Consequently, the model endogenously generates a dual economy with workers whose earnings cyclicality depends on the extensive margin of employment (the lowest deciles of income distribution), and higher income workers whose earnings cyclicality depend on the quality of their employment over time (see Doniger, 2023).

6 What accounts for the increased length of recessions?

The time economies take to recover from recessions has increased across developed economies in the last thirty years. **Figure 15** shows the average number of quarters aggregate GDP has been below trend during recessions for a subset of advanced

Figure 15. Length of Recessions Over Time



Note: The figure reports the average duration of recessions for a set of OECD countries. Specifically, for each recession, we compute the number of quarters each economy's GDP remains below trend.

economies. From the mid-1980s and early 1990s, in particular, this measure has consistently increased.[35] This is consistent with evidence presented by Fukui et al. (2023) on the slower recovery of employment after recessions. Among other factors, an increase in job polarization and the rise in the skill premium (Goldin and Katz 2007; Goos, Manning and Salomons 2009) are contemporaneous phenomena with this rise in the time economies need to recover from recessions. Our model provides a useful framework to check whether human capital accumulation and the sorting dynamics in the labor market contribute to these aggregate developments.

Our model is calibrated to match the differences in career paths and human capital accumulation of graduates and non-graduates in Italy in the last ten years. It indicates that college-educated workers' human capital accumulates faster than other workers', i.e $\phi_g > \phi_s$. The widening of this gap through a higher weight of firm quality in the human capital production function (human capital deepening) proxies the development of a skill premium over time. We conduct the following experiment to analyze whether the rise in human capital's deepening and the lengthening of job ladders have contributed to the increase in length of recoveries. We consider two simulations of the model: our baseline and a counterfactual simulation in which the accumulation of human capital is the same across education levels.[36] In the counterfactual economy the average earnings for graduates are only 1.3% higher than non-graduates.In our baseline economy graduates

[35] We define a recession in the data as occurring after two consecutive quarters of negative GDP growth, and within each recession, we count the number of quarters in which GDP realizations are below trend. We obtain a similar picture if we look at alternative definitions of recession lengths, such as the number of quarters that are needed to reach pre-recession GDP levels and the number of consecutive quarters with negative GDP growth.

[36] We use the same realizations of aggregate productivity shocks for the two experiments, which implicitly assumes that the nature of TFP shocks has not changed over the same time period. This is consistent with Aikman, Drehmann, Juselius and Xing (2022), who show that the source of shocks is not a relevant driver for the severity of downturns, while the magnitude is.

enjoy a faster human capital accumulation, which translates to approximately 38% higher average earnings for graduates.

We report the comparison of these simulations in the first column of **Table 8**. The presence of a high-skill premium leads to recessions that are 1.6 quarters longer than those occurring in the simulation without skill premium. This change in the length of recessions is remarkably close to what is observed for the subset of advanced economies in **Figure 15**. For these countries, in fact, the average duration of below-trend GDP realizations in recessions increases by 1.9 quarters ($\approx$ 29%) comparing the periods before and after 1990.[37]

Jaimovich and Siu (2009) argue that population aging has had a significant impact on the business cycle properties advanced economies. To see how this has affected recession lengths, we simulate the model without a skill premium using demographic weights from the 1970 and 2011 Italian censuses, and we calculate the average recession length in these counterfactual scenarios.[38]

Table 8. Recession lengths

Model	2011 Pop. Weights	1970 Pop. Weights
With skill premium, $\phi_g > \phi_s$	9.6	
Without skill premium, $\phi_g = \phi_s$	8.0	7.7
Data	**Post-1990 Average**	**Pre-1990 Average**
Average (OECD-8)	8.9	7
Italy	15.3	10.7

Note: The table reports the average recession lengths in the data and in the model for different parametrization of the skill premium. In the parametrization without skill premium, we set the elasticity of the human capital accumulation function equal to ϕ_s for both graduates and non-graduates. In the model, we measure the duration of a recession as the average number of consecutive quarters in which the Hamilton filtered GDP (8 leads, 4 lags) is negative.

We find that accounting only for these demographic trends increases the average recession length by 0.3 quarters. Considering both channels simultaneously the model with skill premium and 2011 population weights would deliver recessions approximately two quarters longer than a counterfactual economy without skill-premium and 1970 population weights, the same increase observed, on average, among the OECD economies in **Figure 15**.

[37]For Italy, the increase in this measure has been slightly more pronounced. Recession lengths went from 11 to 15 quarters, approximately a 36% increase.

[38]We also test competing hypotheses, related to concurrent medium-term trends, to show how they relate to recoveries' lengths. Increasing the firm-quality elasticity in the production function by approximately 6%, similar to the change in the share of labor compensation over GDP in Italy after 1990, would not lead to any change in the length of recessions.

7 Conclusions

We develop a novel framework of on-the-job search and human capital accumulation with two-sided heterogeneity to study the role of sorting along the business cycle. In the model, search frictions and aggregate uncertainty prevent an efficient allocation of workers to firms, and the accumulation of human capital depends on the quality of the worker-firm match. Workers direct their search toward less-productive firms in bad times because of lower job finding probabilities, and thus accumulate less human capital. Alterations to the sorting induced by recessions are slow to reverse and contribute not only to slow recoveries but also to long-run changes in the distribution of firms and workers in the labor market. We show that these distortions account for 60% of the cumulative loss in output three years from a recessionary event.

In our model, the impact of downturns is highly heterogeneous across workers. Low-skilled workers are mostly affected by displacement risk, and their earnings have higher volatility. In contrast, high-skilled workers' wages are impacted by the deterioration in sorting - that is, in their ability to climb the higher rungs of the job ladder. The resulting welfare cost of business cycles is heterogeneous with respect to workers' income, human capital and age, and on average is above 2% of consumption. We also show that increases in the importance of on-the-job human capital accumulation, together with aging dynamics, can account for the increased length of recessions in recent decades observed across all developed economies.

The model is rich and tractable, and would naturally lend itself to policy analysis. For instance, the model might be deployed to analyze the impact of policies such as countercyclical unemployment benefits, short-term work and retention schemes, the minimum wage. We leave these studies for future research.

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Online Appendix for
"A Labor Market Sorting Model of Scarring and Hysteresis"

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September 2023

Contents

A Discussion and Proofs

In this section we discuss the properties of the equilibrium of the model economy developed in the previous sections. All propositions and corresponding proofs related to the properties discussed in **Section 3** are reported in **Appendix A.1** and **A.2**.

A.1 Workers optimal behavior

For compactness of notation, we omit the dependence on education level, ι, which is a fixed characteristic, and the idiosyncratic human capital shock, which is additive, from the proofs in Appendices. The logic of the proofs follows without loss of generality.

The following propositions characterize the properties of workers' optimal search strategies that solve the search problem in (9), restated here for convenience:

$$R(h, \tau, V; \Omega) = \sup_{v} \Big[p(\theta(h, \tau, v; \Omega)) \big[v - V \big] \Big]. \tag{A.1}$$

Lemma A.1. *The composite function $p(\theta(h,\tau,v;\Omega))$ is strictly decreasing and strictly concave in v.*

Proof. For this proof we follow closely Menzio and Shi (2010), Lemma 4.1 (ii). From the properties of the matching function we know that $p(\theta)$ is increasing and concave in θ, while $q(\theta)$ is decreasing and convex. Consider that the equilibrium definition of $\theta(\cdot)$ is

$$\theta(h,\tau,v;\Omega) = q^{-1}\left(\frac{c(y)}{J(h,\tau,v,y;\Omega)}\right),$$

and that the first order condition for the wage and the envelope condition on V of the optimal contract problem in (5) implies

$$\frac{\partial J(h,\tau,v,y;\Omega)}{\partial v} = -\frac{1}{u'(w)}.$$

so that as $u'(\cdot) > 0$, $J(\cdot)$ is decreasing in v.

From the equilibrium definition of $\theta(\cdot)$ and noting that $q^{-1}(\cdot)$ is also decreasing due to the properties of the matching function we have that

$$\frac{\partial\theta(h,\tau,v;\Omega)}{\partial v} = \left.\frac{\partial q^{-1}(\xi)}{\partial \xi}\right|_{\xi=\frac{c(y)}{J(h,\tau,v,y;\Omega)}} \cdot \left(-\frac{\partial J(h,\tau,v,y;\Omega)}{\partial v}\right) \cdot \frac{c(y)}{(J(h,\tau,v,y;\Omega))^2} < 0,$$

which, in turn, implies that

$$\frac{\partial p(\theta(h,\tau,v;\Omega))}{\partial v} = \left.\frac{\partial p(\theta)}{\partial\theta}\right|_{\theta=\theta(h,\tau,v;\Omega)} \cdot \frac{\partial\theta(h,\tau,v;\Omega)}{\partial v} < 0.$$

Suppressing dependence on the states (h,τ,y,Ω) for readability, to prove that $p(\theta(v))$ is concave, consider that $J(v)$ is concave[1] and a generic function $\frac{c}{v}$ is strictly convex in v. This implies that with $\alpha \in [0,1]$ and $v_1, v_2 \in \mathcal{V}$, $v_1 \neq v_2$:

$$\frac{c}{J(\alpha v_1 + (1-\alpha)v_2)} \leq \frac{c}{\alpha J(v_1) + (1-\alpha)J(v_2)} < \alpha\frac{c}{J(v_1)} + (1-\alpha)\frac{c}{J(v_2)}.$$

As $p(q^{-1}(\cdot))$ is strictly decreasing the inequality implies that

$$\begin{aligned} p\left(q^{-1}\left(\tfrac{c}{J(\alpha v_1+(1-\alpha)v_2)}\right)\right) &\geq p\left(q^{-1}\left(\tfrac{c}{\alpha J(v_1)+(1-\alpha)J(v_2)}\right)\right) \\ &> \alpha p\left(q^{-1}\left(\tfrac{c}{J(v_1)}\right)\right) + (1-\alpha)p\left(q^{-1}\left(\tfrac{c}{J(v_2)}\right)\right), \end{aligned}$$

[1] $J(\cdot)$ concave give the two-point lottery in the structure of the contract. See Menzio and Shi (2010) Lemma F.1.

and as $\theta(v) = q^{-1}(\frac{c}{J(v)})$:

$$p(\theta(\alpha v_1 + (1-\alpha)v_2)) > \alpha p(\theta(v_1)) + (1-\alpha)p(\theta(v_2))$$

so that $p(\theta(v))$ is strictly concave in v. $\square$

In the following proposition we summarize the main results regarding the behavior of the workers and their objective functions.

Proposition A.1. *Given the worker search problem, the following properties hold:*

(i) The returns to search, $p(\theta(h,\tau,v;\Omega))\big[v-V\big]$, are strictly concave with respect to promised utility, v.

(ii) The optimal search strategy

$$v^*(h,\tau,V;\Omega) \in \arg\max_v \big\{p(\theta(h,\tau,v;\Omega))\big[v-V\big]\big\}$$

is weakly increasing in V, Liptschitz continuous and unique.

(iii) For all promised utilities, the search gain $R(h,\tau,V;\Omega)$ is positive, weakly decreasing in V.

(iv) The survival probability of the match, given the optimal choice of the worker, is increasing in the value of current and future promised utilities, so $\widetilde{p}_t(h,\tau,v;\Omega)$ is increasing in v and V.

(v) Search strategies are increasing in workers' human capital, h.

Proof. The proofs follow closely Shi (2009), Lemma 3.1 and Menzio and Shi (2010), Corollary 4.4. More formally, for each triplet (h,τ,Ω) given at each search stage, we can re-define the search objective function as $K(v,V) = p(\theta(v))(v-V)$ and $v^*(V) \in \arg\max_v K(v,V)$ as the function that maximises the search returns (i.e. the optimal search strategy of the worker) and prove the following

(i) To show that $K(v,V)$ is strictly concave in v consider two values for v, $v_1, v_2 \in \mathcal{V}$ such that $v_2 > v_1$ and define $v_\alpha = \alpha v_1 + (1-\alpha)v_2$ for $\alpha \in [0,1]$.

Then by definition:

$$\begin{aligned} K(v_\alpha, V) = {} & p(\theta(v_\alpha))(v_\alpha - V) \\ & \geq [\alpha p(\theta(v_1)) + (1-\alpha)p(\theta(v_2))][\alpha(v_1 - V) + (1-\alpha)(v_2 - V)] \\ & = \alpha K(v_1, V) + (1-\alpha)K(v_2,V) + \alpha(1-\alpha)[(p(\theta(v_1)) - p(\theta(v_2))](v_2 - v_1) \\ & > \alpha K(v_1,V) + (1-\alpha)K(v_2,V) \end{aligned}$$

where the first inequality follows from the concavity of $p(\theta(\cdot))$ (this is true if $J(\cdot)$ concave with respect to V) and the second inequality stems from the fact that $p(\theta(\cdot))$ is strictly decreasing hence $\alpha(1-\alpha)[(p(\theta(v_1)) - p(\theta(v_2))](v_2 - v_1) > 0$.

(ii) **Weakly Increasing.** Consider a worker employed in a job that gives lifetime utility V. Given that $v \in [\underline{v}, \overline{v}]$, and that submarkets are going to open depending on realizations of the aggregate productivity, a, there is only one region in the set of promised utilities where the search gain is positive. This set is $[V, v(a)]$ with $v(a)$ being the highest possible offer that a firm makes in the submarket for the worker (h, τ). Any submarket that promises higher than $v(a)$ is going to have zero tightness. Therefore, the optimal search strategy for $V \geq v(a)$ is $v^*(V) = V$, as $K(V, v(a)) = K(V, V) = K(\overline{v}, V) = 0$ (the search gain is null given the current lifetime utility V). For $V \in [V, v(a)]$, instead, as $K(v, V)$ is bounded and continuous, the solution $v^*(V)$ has to be interior and therefore respect the following first order condition

$$V = v^*(V) + \frac{p(\theta(v^*(V)))}{p'(\theta(v^*(V)) \cdot \theta'(v^*(V))}. \tag{A.2}$$

Now consider two arbitrary values V_1 and V_2, $V_1 < V_2 < \overline{v}$ and their associated solutions $W_i = v^*(V_i)$ for $i = 1, 2$. Then, V_1 and V_2 have to generate two different values for the right-hand side of (A.2). Hence, $v^*(V_1) \cap v^*(V_2) = \emptyset$ when $V_1 \neq V_2$.

This also implies that the search gain evaluated at the optimal search strategy is higher than the gain at any other arbitrary strategy so that $K(W_i, V_i) > K(W_j, V_i)$ for $i \neq j$. This implies that

$$\begin{aligned} 0 \geq & [K(W_2, V_1) - K(W_1, V_1)] + [K(W_1, V_2) - K(W_2, V_2)] \\ & = (p(\theta(W_2)) - p(\theta(W_1)))(V_2 - V_1), \end{aligned}$$

thus, $p(\theta(W_2)) \leq p(\theta(W_1))$. As $p(\theta(\cdot))$ is strictly decreasing (see Lemma A.1), then $v^*(V_1) \leq v^*(V_2)$.

Lipschitz continuous. We follow Menzio and Shi (2010) and show that, with generic points $V_1 \leq V_2$, $v^*(V_2) - v^*(V_1) \leq V_2 - V_1$. Given that in this case we know that $v^*(V_1) \leq v^*(V_2)$, then $v^*(V_2) - v^*(V_1) \geq 0$. If $v^*(V_2) - v^*(V_1) = 0$ then the claim is trivially satisfied. If $v^*(V_2) - v^*(V_1) > 0$, then let's consider a generic real number $\delta \in (0, (v^*(V_2) - v^*(V_1))/2)$. Given the definition of $v^*(\dot{})$, we know that $K(v^*(V_i), V_i) \geq K(v^*(V_i) + \delta, V_i)$. From this inequality and the definition for $K(\cdot, \cdot)$ it follows that $p(\theta(v^*(V_1)))(v^*(V_1) - V_1) \geq p(\theta(v^*(V_1) + \delta))(v^*(V_1) + \delta - V_1)$ and similarly $p(\theta(v^*(V_2)))(v^*(V_2)-V_2) \geq p(\theta(v^*(V_2)-\delta))(v^*(V_2)\delta - V_2)$. Then

$$\begin{aligned} v^*(V_1)-V_1 &\geq \frac{p(\theta(v^*(V_1)+\delta))\delta}{p(\theta(v^*(V_1)))-p(\theta(v^*(V_1)+\delta))} \\ v^*(V_2)-V_2 &\leq \frac{p(\theta(v^*(V_2)-\delta))\delta}{p(\theta(v^*(V_2)-\delta))-p(\theta(v^*(V_2)))}. \end{aligned}$$

Knowing that $p(\theta(\cdot))$ is a decreasing function, and that $v^*(V_1)+\delta \leq v^*(V_2)-\delta$ given the domain of δ, then $p(\theta(v^*(V_1)+\delta) \geq p(\theta(v^*(V_2)-\delta)$. Also, given that $v^*(V_2)-v^*(V_1) > 0$ then $p(\theta(v^*(V_1)))-p(\theta(v^*(V_1)+\delta)) \leq p(\theta(v^*(V_2)-\delta))-p(\theta(v^*(V_2)))$. This implies that we can rewrite the inequalities above as

$$\begin{aligned} v^*(V_2)-V_2 &\leq \frac{p(\theta(v^*(V_2)-\delta))\delta}{p(\theta(v^*(V_2)-\delta))-p(\theta(v^*(V_2)))} \\ &\leq \frac{p(\theta(v^*(V_1)+\delta)}{p(\theta(v^*(V_2)-\delta))-p(\theta(v^*(V_2)))} \\ &\leq \frac{p(\theta(v^*(V_1)+\delta)}{p(\theta(v^*(V_1)))-p(\theta(v^*(V_1)+\delta))} \leq v^*(V_1)-V_1 \end{aligned}$$

which implies $v^*(V_2)-v^*(V_1) \leq V_2 - V_1$.

Unique. Uniqueness follows directly from strict concavity shown in (i).

(iii) The Bellman equation for the search problem is:

$$R(h,\tau,V;\Omega) = \sup_v \Big[p(\theta(h,\tau,v;\Omega))\big[v-V\big]\Big]$$

hence a simple envelope argument shows that

$$\frac{\partial R(h,\tau,V;\Omega)}{\partial V} = -p(\theta(h,\tau,v;\Omega)) \leq 0,$$

as the job finding probability is weakly positive for all utility promises.

As $p(\theta(\cdot)) \geq 0$, $v^*(\cdot) \in [\underline{v}, \overline{v}]$ then $R(\cdot) \geq 0$.

(iv) Given the optimal search strategy, $v^*(h,\tau,V;\Omega)$, we can define the survival probability of the match as in (12):

$$\widetilde{p}(h,\tau,V;\Omega) \equiv (1-\lambda)(1-\lambda_e p(\theta(h,\tau,v^*;\Omega))).$$

Then, given (h,τ,Ω)

$$\frac{\partial \widetilde{p}(V)}{\partial V} = -\beta(1-\lambda)\lambda_e \left.\frac{\partial p(\theta)}{\partial \theta}\right|_{\theta=\theta(v^*)} \left.\frac{\partial \theta(v)}{\partial v}\right|_{v=v^*(V)} \frac{\partial v^*(V)}{\partial V} > 0,$$

because $p(\cdot)$ and $v^*(\cdot)$ are both increasing functions while $\theta(\cdot)$ is a decreasing function in promised utilities.

(v) We want to show that the optimal search strategy v^*, is increasing in human capital, so that $\frac{\partial v^*(h)}{\partial h} > 0$.

Knowing that the optimal search strategy has to satisfy the following first order condition:

$$\frac{\partial p}{\partial v}(v^* - V) = 0,$$

we can use the implicit function theorem to get

$$\frac{\partial v^*}{\partial h} = -\frac{(v^* - V)\frac{\partial^2 p}{\partial v \partial h} + \frac{\partial p}{\partial h}}{(v^* - V)\frac{\partial^2 p}{\partial v^2} + 2\frac{\partial p}{\partial v}}. \tag{A.3}$$

As $(v^* - V) > 0$, and $p(\cdot)$ decreasing and concave in W so that $\frac{\partial^2 p}{\partial W^2}, \frac{\partial p}{\partial W} < 0$, the denominator of **Equation A.3** is negative. Therefore for **Equation A.3** to be negative, it has to be that $(v^* - V)\frac{\partial^2 p}{\partial W \partial h} + \frac{\partial p}{\partial h} > 0$. This, in turn, implies that

$$\frac{\partial^2 p}{\partial W \partial h} > \frac{\partial p}{\partial W}\frac{\partial p}{\partial h}\frac{1}{p},$$

where the left hand side is negative as $p(\cdot)$ is increasing in h but decreasing in W.

□

The first statement implies that the marginal returns of searching towards better firms are decreasing. The intuition is that as workers search for work at firms granting better values, their job-finding probability decreases as better employment prospects are also subject to higher competition.

As a consequence of the strict concavity established in the first statement, workers' optimal search strategy is unique. The search strategy is also (weakly) increasing in the value of lifetime utilities granted by the current contract, which is the outside option for the worker.

The third statement follows from the fact that marginal returns to search are decreasing and the set of feasible utility promises is compact. The intuition is that employees at firms with higher utility promises have a relatively fewer chances of improving their position. Given a high outside option, the utility gain from moving is relatively lower, whereas the probability of matching with any firm does not depend on the *current* utility promise per se, but on the future promise offered by the vacancy.

The fourth statement finally follows from considering the implication of the previous ones. Given that the optimal search strategy is increasing in V workers' probability of leaving the firm at any time ends up depending negatively on V. This guarantees a longer expected duration of the match at higher current promised utility V, thus retention probabilities that are increasing in promised utilities v.

As human capital accumulation is tightly linked to the quality of the employer, workers that are able to start their working careers in good times have a greater chance of finding themselves on an higher path of human capital growth. As worker careers are limited and human capital accumulation follows a slow-moving process, business cycle effects on human capital quality fade only slowly and the quality of initial matches, both in terms of lifetime utility and firm quality, bears a long-standing effect on workers' careers.

A.2 Characteristics of the optimal contract

The optimization in the contracting problem balances a trade-off between insurance provision and profit maximization for firms. The contract implicitly takes into account workers' search incentives and their inability to commit to stay. The following propositions characterizes workers' incentives along the business cycle from the firms' standpoint.

Lemma A.2. *The Pareto frontier $J(h, \tau, W, y; \Omega)$ is concave in W.*

Proof. This is a direct consequence of using a two-point lottery for $\{w_i, W_i'\}$ as shown by Menzio and Shi (2010), Lemma F.1. □

Lemma A.3. *The Pareto frontier $J(h, \tau, W, y; \Omega)$ is increasing in y.*

Proof. The intuition for this proof follows the fact that a higher y firm, once the match exists, can always deliver a certain promise V and have resources left over. Within a dynamic contract, future retention is already optimized as the match is formed. This means that the promise V can be delivered by the greater capacity on the part of producing with respect to a close y firm. In presence of human capital accumulation, the worker is compensated through greater option values in the future, which again means that, even with lower retention, the firm cashes in more profits while decreasing wages (and respecting the V promise).

One can get to the same conclusion by starting from time T, noticing that the function J is trivially increasing in y in the last period, and the stepping back. At $T-1$, given V, any higher y function can make greater profits with the same delivery of value V, given the contract's optimal promise, which is a fortiori true with human capital accumulation (the option value is greater, so the firm can decrease w as a response). A more formal argument goes as follows: start from the optimal policies, as per Eq. (5) of a firm that has $y = \bar{y}$ and assume that one could exogenously increase its installed capital to $\bar{y} + \varepsilon$. We want to know whether, keeping policies constant, this would increase the flow of profits while keeping the worker indifferent. If this is true, then *a fortiori* it will be true that the firm value function J will be increasing in y. With a slight abuse of notation and for conciseness, we refer to the future J in Eq. (5) as J'.

This amounts to calculating:[2]

$$\left.\frac{d\bar{J}}{dy}\right|_{W,\{\pi_i,w_i,W_i'\}} = \frac{\partial f(\cdot)}{\partial y} + \beta \mathbb{E}_\Omega \left[\left.\frac{\partial \tilde{p}(\cdot)}{\partial y}\right|_{W,\{\pi_i,w_i,W_i'\}} \bar{J}' \right]$$

This first order condition presents the trade-off discussed in words above, namely that an increase in y will be instantaneously beneficial to production, but might also potentially have a longer term adverse effect on profits through decreased retention. Finding the sign of the derivative on the LHS hinges on understanding the sign of the derivative of the second element on the RHS, since $\frac{\partial f(\cdot)}{\partial y} > 0$ given the properties of the production function $f(\cdot)$. The change in y would affect search objective v_y through the variation in h due to the human capital accumulation dynamics, even taking the current firm policies as given.

Notice that:

$$\frac{\partial \tilde{p}(\cdot)}{\partial y} = -\lambda\lambda_E \frac{\partial p(\cdot)}{\partial \theta}\frac{\partial \theta(\cdot)}{\partial y}$$

Remember that $\frac{\partial p(\cdot)}{\partial \theta} > 0$ due to the properties of the job-finding probability function $p(\cdot)$. Assume first the case in which $\frac{\partial \theta(\cdot)}{\partial y} \leq 0$. In this case $\frac{\partial \tilde{p}(\cdot)}{\partial y} > 0$, which in turn implies, as argued, that J has to be increasing in y.

Now consider the second case, namely $\frac{\partial \theta(\cdot)}{\partial y} > 0$. By the free entry condition, we obtain:

$$\frac{\partial \theta(\cdot)}{\partial y} = \frac{\partial q^{-1}(c(y)/J(y))}{\partial y} = \frac{1}{q'\left(q^{-1}\left(c(y)/J(y)\right)\right)} \frac{c'(y)J(y) - \frac{\partial J(y)}{\partial y}c(y)}{J(y)^2}$$

The first term in the result is negative, given the properties of function $q(\cdot)$. Given our assumption on $\frac{\partial \theta(\cdot)}{\partial y}$ the second term in the result has to be negative as well. This requires: $c'(y)J(y) - \frac{\partial J(y)}{\partial y}c(y) < 0$, or $\frac{\partial J(y)}{\partial y} > \frac{c'(y)J(y)}{c(y)} > 0$. $\square$

Proposition A.2. *The Pareto frontier $J(h, \tau, W, y; a, \mu)$ is increasing in the aggregate productivity shock a, while retention probabilities, $\widetilde{p}(h, \tau, W; a, \mu)$ decrease in aggregate productivity.*

[2] Without loss of generality, we assume that J' is constant with respect to y. Alternatively, one may start proving the result for contracts offered to workers just one period before retirement (for which $\frac{\partial J'}{\partial y}$ is trivially positive), and generalize the result with backward induction.

Proof. We proceed by backward induction.[3] Following the logic of the proof of **Lemma A.3**, the proposition is trivially true for workers T periods old. Given that the firm increases its production while keeping the worker at least indifferent, J is at least weakly increasing in a. However, the firm can also feasibly increase the worker's wage by ε, with $\varepsilon < \frac{\partial f(\cdot)}{\partial a}$. J is thus strictly increasing in y.

Consider now a worker who is $T-1$ periods old. A firm matched to a worker in submarket $\{h, T-1, y, W\}$ will face the following Pareto frontier

$$\begin{aligned} J(h, T-1, y, W_y; a, \mu) = \sup_{w, W'} \Big(& f(h, y; a) - w \\ & + \mathbb{E}_\Omega \left[\widetilde{p}(h', T, W'; a', \mu')(f(h', y; a') - w') \right] \Big) \end{aligned}$$

Analogously to the proof in **Lemma A.3**, assume that aggregate productivity increases from $\bar{a}$ to $\bar{a} + \varepsilon$. Assume that the firm keeps its policies constant once again. We aim at proving that, even in such a case, firm value increases while keeping the worker at least indifferent. If this is the case, it is *a fortiori* true that J increases in a after reoptimizing firms' policies.

We are now interested in the sign of:[4]

$$\left.\frac{\partial \bar{J}}{\partial a}\right|_{W, \pi, w, \{W'\}} = \frac{\partial f(\cdot)}{\partial a} + \beta \mathbb{E}_\Omega \left[\left.\frac{\partial \tilde{p}(\cdot)}{\partial a}\right|_{W, \pi_i, w, \{W'\}} \bar{J}' \right]$$

Now notice that, in equilibrium,

$$\frac{\partial \widetilde{p}(\theta)}{\partial a} \propto -\frac{\partial p(\theta)}{\partial a}, \quad \text{with} \quad \frac{\partial p(\theta)}{\partial a} = \underbrace{\frac{\partial p(\theta)}{\partial \theta}}_{>0} \cdot \underbrace{\frac{\partial \theta}{\partial J(\cdot)}}_{>0} \cdot \frac{\partial J(\cdot)}{\partial a}$$

where the sign of the second derivative on the right hand side comes from the free entry condition and the properties of vacancy filling probability function $q(\cdot)$. Given this, it has to be that $\frac{\partial p(\theta)}{\partial a}$ and $\frac{\partial J(\cdot)}{\partial a}$ have the same sign in equilibrium. Now, if both are strictly positive, both statements of our proposition are immediately true. Let's now assume they are both negative or zero. If this is the case, then $\frac{\partial \tilde{p}(\cdot)}{\partial a} \geq 0$. But this implies $\frac{\partial \bar{J}}{\partial a} > 0$, which is a contradiction. □

The intuition behind this proposition relies on the observation that higher productiv-

[3] For compactness of notation, we omit without loss of generality the two-point lottery in the equations in the proof.

[4] We assume that $J' = f(h', y; a') - w'$ is constant with respect to a. It is possible to prove, by backward induction, that this assumption is without loss of generality for the sake of the proof.

ity realization are associated not only with better outcomes on impact but also to better future prospects, given that the productivity process is an increasing Markov chain.

A key property of the model is that it allows to characterize the workers' optimal behaviour along the business cycle. The following proposition summarizes how the search strategy changes depending on the aggregate productivity realization.

Proposition A.2 and Corollary A.3 have an important implication regarding firms' vacancy posting and workers' search decisions. The fact that at the posting stage profits J are increasing in aggregate productivity implies that more entry will take place in good times, and ceteris paribus more entrepreneurs will open up vacancies across the whole firms' distribution.[5] The resulting higher tightness impacts workers' optimal search behaviour as the job finding probability increases in all submarkets. As a consequence, workers respond optimally to the productivity increase searching in submarkets that guarantee higher lifetime utility promises.

Firms utility promises depend on the structure of the optimal contract. The contract provides insurance to workers through wage paths that are downward rigid, and at the same time allows firms to profit as wages only partially adjust to productivity realizations.

The following propositions provide a clear picture of the growth path prescribed by the optimal contract for a continuing firm. First, let us define the productivity threshold that determines whether a worker-firm match does not survive.

Corollary A.1. *There exists a productivity threshold $a^*(h, \tau, W, y)$ below which firms will not continue the operate.*

Proof. The proof follows immediately from **Proposition A.2** and the timing of the shock. Given the timing of the shock, exit is fully determined by the current productivity shock and incumbent firms know in advance whether they are willing to produce in the next period.

Therefore, as the Pareto frontier is strictly increasing in a, firms are willing to continue the contract if $\mathbb{E}_\Omega[J(h', \tau + 1, W', y; a', \mu')|h, \tau, W, y, a, \mu] \geq 0$, so that the threshold that determines exit is

$$a^*(h, \tau, W, y) : \mathbb{E}_\Omega[J(h', \tau + 1, W', y; a', \mu')|h, \tau, W, y, a, \mu] = 0.$$

□

The intuition of why this has to be the case is linked to the fact that the Pareto frontier is strictly increasing in a and decreasing in the level of promised utilities to the

[5] In our model a better firm is a more productive firm. We do not specifically model the determinant of quality heterogeneity but we take the existence of profound differences in firm quality as a fact (Arellano-Bover, 2020, Arellano-Bover and Saltiel, 2022).

worker. Hence once the aggregate state realizes a firm is able to perfectly predict whether next period it will exit the market or stay in (given the timing, the decision is based on expected profits, and is thus *not* state-contingent to next period's productivity). The choice is taken *before* new realizations of productivity, so it is possible that a firm makes negative profits for at most one period.

Corollary A.2. *The productivity threshold $a^*(h,\tau,y,W)$ below which firm y in match with worker (h,τ) and promised utility W exits the market in the aggregate state Ω is increasing in y.*

Proof. Consider two firms characterized by y_1, y_2 with $y_1 < y_2$. Consider the threshold for firm y_1, $a_1^* = a^*(h,\tau,,W_{y_1},y_1)$. Firm y_1 makes 0 profits if state a_1^* materializes next period. Consider firm y_2 trying to mimic the current contract offered by y_1 to (h,τ). We know that J is increasing in y from **Lemma A.3**, which implies that the firm is making a profit at a_1^*. This completes the proof. $\square$

Lemma A.4. *The Pareto frontier $J(h,\tau,W,y;\Omega)$ is strictly concave in y.*

Proof. **No human capital accumulation.** The proof without human capital accumulation is straightforward. Assuming there is no dependency of h' on y, one can write:

$$\frac{dJ}{dy} = f_y + \widetilde{p}\frac{dJ'}{dy}$$

$$\frac{d^2J}{dy^2} = f_{yy} + \widetilde{p}\frac{d^2J'}{dy^2}$$

where we take J' to represent next period's firm value $J(\cdot)$, and the dependence of all controls on y is ignored by virtue of the envelope condition. One can readily observe by induction that, given the concavity of $f(\cdot)$ in y, $\frac{d^2J}{dy^2} < 0$.

Human capital accumulation. With human capital accumulation the concavity of the function J on y depends on further assumptions on the concavity of the human capital accumulation function $g(\cdot)$. As these assumptions involve equilibrium objects, we state them here and verify ex-post numerically that they are always respected in our setting.

$$\frac{dJ}{dy} = f_y + g_y\frac{\partial\widetilde{p}J'}{\partial h} + \widetilde{p}\frac{dJ'}{dy}$$

$$\frac{d^2J}{dy^2} = f_{yy} + \underbrace{g_{yy}}_{<0}\frac{\partial\widetilde{p}J'}{\partial h'} + g_y^2\frac{\partial^2\widetilde{p}J'}{\partial h'^2} + \underbrace{2g_y}_{>0}\left(\frac{\partial\widetilde{p}}{\partial h'}f_y + \widetilde{p}f_{hy}\right) + \widetilde{p}\frac{d^2J'}{dy^2}$$

Sufficient conditions for the previous expression to be true at $T-1$ would jointly be:

$$\tilde{p}_h(f-w) + f_h\tilde{p} \geq 0 \tag{A.4}$$

$$\tilde{p}_h f_y + \tilde{p} f_{h,y} \leq 0 \tag{A.5}$$

$$\tilde{p}_{h,h}(f - w) + \tilde{p} f_{h,h} + 2\tilde{p}_h f_h \leq 0 \tag{A.6}$$

Rearranging and combining the first two would lead to:

$$(f - w) \leq \frac{f_y f_h}{f_{h,y}} \Rightarrow f \leq \frac{f_y f_h}{f_{h,y}} + w \tag{A.7}$$

Since $w > 0$, a more stringent condition would be:

$$\frac{f f_{h,y}}{f_y f_h} \leq 1 \rightarrow ES \leq 1 \tag{A.8}$$

This condition is not generally too restrictive, and holds for instance with equality in the Cobb-Douglas case.

Going back to the third condition, and combining it with the first one, one gets

$$\frac{\tilde{p}}{\tilde{p}_h}\left(f_{h,h} - \frac{\tilde{p}_{h,h}}{\tilde{p}_h} f_h\right) + 2 f_h \geq 0 \tag{A.9}$$

A sufficient condition for that would be

$$h\frac{f_{h,h}}{f_h} \leq h\frac{\tilde{p}_{h,h}}{\tilde{p}_h} \tag{A.10}$$

In the Cobb-Douglas case this condition amounts to

$$\alpha \geq -h\frac{\tilde{p}_{h,h}}{\tilde{p}_h} \tag{A.11}$$

which bounds the returns on human capital. The economic interpretation is simple. For J to be concave even with endogenously increasing human capital, it has to be the case that the marginal product of human capital is not too big in the production function. For this to be the case, the effect of the relative flattening of the production function and decrease in marginal product of human capital has to be dominated by the relative increase in the marginal effect of the decrease in retention.

□

Proposition A.3. *For each state in which the firm is willing to continue the contract, the optimal contract delivers a wage path that follows firms profits according to the wage Euler equation:*

$$\frac{\partial \widetilde{p}(\Theta)}{\partial W_i'} \frac{J'(\Theta)}{\widetilde{p}(\Theta)} = \frac{1}{u'(w_i)} - \frac{1}{u'(w)} \tag{A.12}$$

with $\Theta \equiv (\phi(h, y), \tau + 1, W_i'; \Omega')$ *being the definition of the relevant state and* w_i *is the* *wage paid in the future state.*

Proof. Consider the firm problem in Equation (5), restated here for convenience

$$J(h,\tau,W,y;\Omega) = \sup_{\{\pi_i,w_i,W_i\}} \sum_{i=1,2} \pi_i \Big(f(y,h;\Omega) - w_i \\ + \mathbb{E}_\Omega \left[\widetilde{p}(h',\tau+1,W_i;\Omega') J(h',\tau+1,W_i,y;\Omega') \right] \Big)$$

$$s.t.\ [\lambda]\ W = \sum_{i=1,2} \pi_i \left(u(w_i) + \mathbb{E}_\Omega \widetilde{r}(h',\tau+1,W_i;\Omega') \right),$$
$$\sum_{i=1,2} \pi_i = 1, \quad h' = \phi(h,y).$$

For $i = 1,2$, the first order conditions with respect to the wage and the promised utilities are:

$$[w_i]:\ \lambda = \frac{1}{u'(w_i)} \tag{A.13}$$
$$[W_i]:\ \frac{\partial \widetilde{p}(\cdot)}{\partial W_i} J(\cdot) + \widetilde{p}(\cdot) \frac{\partial J(\cdot)}{\partial W_i} + \lambda \frac{\partial \widetilde{r}(\cdot)}{\partial W_i} = 0. \tag{A.14}$$

Note that by definition,

$$\widetilde{r}(h,\tau,V;\Omega) \equiv \lambda U(h,\tau;\Omega) + (1-\lambda)\Big[W + \lambda_e \max\{0, R(h,\tau,V;\Omega)\} \Big]$$

therefore we can use the envelope theorem as in Benveniste and Scheinkman (1979), Theorem 1 and the definition in Equation (12) to derive an expression for the derivative of the employment value in $t+1$ as the period ahead of the following:

$$\frac{\partial \widetilde{r}(h,\tau,W;\Omega)}{\partial W} = \widetilde{p}(h,\tau,W;\Omega).$$

Similarly, using the envelope condition on the firm problem and the first order condition for the wage, we can establish that

$$\frac{\partial J(h,\tau,y,W;\Omega))}{\partial W} = -\lambda \ \therefore\ \frac{\partial J(h,\tau,W,y;\Omega))}{\partial W} = -\frac{1}{u'(w)}. \tag{A.15}$$

Moving these two expressions one period ahead, substituting them in (A.14), focusing on $\widetilde{p(\cdot)} > 0$ and $\pi_i > 0$ and rearranging we have that:

$$\frac{\partial \widetilde{p}(\Theta)}{\partial W_i} \frac{J(\Theta)}{\widetilde{p}(\Theta)} = \frac{1}{u'(w_i')} - \frac{1}{u'(w)},$$

with $\Theta \equiv (\phi(h,y), \tau+1, W; \Omega')$ and where w' is the wage next period in state Ω'. □

The optimal contract links the wage growth to the realization of firms profits. The right hand side of Equation A.12 shows that, in providing insurance to the worker, the firm links wage growth to profits and to the incentive to maximize retention, incorporated in $\frac{\partial \log \widetilde{p}}{\partial W}$, the semi-elasticity of the retention probability to the utility offer. As the production stage takes place *after* exit choices are taken by the incumbent firms, the wage growth related to the continuation value of the contract is bound to be (weakly) positive, hence workers enjoy a non-decreasing wage profile under the optimal contract.[6]

A feature that the optimal contract derived in our model shares with the literature on long-term contracts with lack of commitment on the worker side is thus the backloading of wages.[7] Workers in our model make search decisions that affect the survival probability of the match. They do not however appropriate the full future value of the current match while making these search decisions (unless the firm makes zero profits). This makes it optimal for the firm to front-load profits and back-load wages. The reason is that the firm provides insurance and income smoothing to the worker, but given its risk neutrality it prefers to front-load its profits while providing an increasing compensation path to maximize retention. The contract thus optimally balances the consumption smoothing motives (i.e. the insurance provision of the contract) with the commitment problem of the worker.

Special case with log-utility. The wage Euler equation discussed in **Proposition A.3** can be simplified to a more intuitive interpretation in the log-utility case. In case of log-utility, in fact, $u'(w_{i,\Omega}) = \frac{1}{w_{i,\Omega}}$. Multiplying and dividing by wage levels and rearranging, we can express the elasticity of retention probability to offered utility as

$$\varepsilon_{\widetilde{p},W_y} = \underbrace{\frac{(w_i - w)}{w}}_{\text{Wage growth}} \quad \underbrace{\frac{w}{J(\Theta)}}_{\substack{\text{Ratio of wage}\\ \text{to match value}}} \quad . \tag{A.16}$$

with $\varepsilon_{\widetilde{p},W} \equiv \frac{\partial \widetilde{p}(\Theta)}{\partial W_i \widetilde{p}(\Theta)}$.

The interpretation of this result is of interest to analyses that relate labor market dynamism to wage dynamics, like Engbom (2021). This is because $\varepsilon_{\widetilde{p},W_y}$, being a function of structural parameters of the matching technology, γ, search frictions λ_e, and measures of labor market tightness θ, provides us with a good proxy of labor market fluidity. The right hand side of (A.16), is composed entirely of observable quantities, as the ratio of wages to match value is a function of factor shares in value added. The quantity can then be used to compare the dynamism of different local, regional or national labor markets.

[6]As the exit decision takes place by considering *expected* profits next period, a firm operating at low but positive expected profits might end up, at most for a period, to have a negative continuation value. This would imply that wage growth *can* be negative before a firm's closure, which is actually a common finding in empirical studies (firstly observed in Ashenfelter (1978)).

[7]See for instance, Thomas and Worral (1988), Tsuyuhara (2016) and Balke and Lamadon (2022).

The next proposition, instead, confirms our initial conjecture that in equilibrium firm qualities and utility promises are related to a one-to-one mapping.

Lemma A.5. $\Pi(y,h,\tau,W;\Omega)$ *is concave in* W *and optimal utility promises are unique and increasing in* y *given worker characteristics* (h,τ) *and aggregate state* Ω.

Proof. Assuming the same (h,τ,y), the entrepreneur then chooses the optimal value $W^*(y) := \arg\max_W \Pi(W;\Omega)$ to deliver in the contract. For the rest of the proof we consider as given the dependence of the functions on (h,τ,Ω) and consider directly the composite function $q(\theta(W))$ as $q(W)$.

The optimization requires that W^* satisfies the following first order condition:

$$q_W(W^*)J(y,W^*) + q(W^*)J_W(y,W^*) = 0. \tag{A.17}$$

For W^* to be a unique maximum, $\Pi(\cdot)$ has to be concave hence the second-order condition must be negative. Taking into account explicitly of the composite function $q(\theta(W))$ implies the following second-order condition:

$$q_{\theta,\theta}(\theta_W)^2 J + q_\theta \theta_{W,W} J + 2q_\theta\theta_W J_W + qJ_{W,W} < 0 \tag{A.18}$$

$$(q_{\theta,\theta}\theta_W + q_\theta\frac{\theta_{W,W}}{\theta_W})\theta_W J + 2q_\theta\theta_W J_W + qJ_{W,W} < 0 \tag{A.19}$$

$$(q_{\theta,\theta}\theta_W + q_\theta\frac{\theta_{W,W}}{\theta_W})\underbrace{\theta_W J}_{<0} + \underbrace{2q_\theta\theta_W J_W + qJ_{W,W}}_{<0} < 0 \tag{A.20}$$

where the inequalities in **Equation A.20** follow from knowing that $J(\cdot)$ is concave and decreasing in W, $J_W, J_{W,W} \leq 0$, $q(\cdot)$ is increasing in W, and $\theta(\cdot)$ is decreasing in promises as discussed in **Lemma A.1**. Note that if $q_{\theta,\theta}\theta_W + q_\theta\frac{\theta_{W,W}}{\theta_W} > 0$ then the inequality in A.20 would be respected and $\Pi(\cdot)$ would be concave. For this term to be positive it has to be that:

$$\frac{q_{\theta,\theta}}{q_\theta}\theta_W > -\frac{\theta_{W,W}}{\theta_W}. \tag{A.21}$$

As $q(\cdot)$ is decreasing and convex, $q_{\theta,\theta} > 0, q_\theta < 0$ and $\theta_W < 0$, the left-hand side of A.21 is positive. If $\theta_{W,W} < 0$ the right-hand side would be negative and the inequality would always be respected. We know from **Lemma A.1** that $\theta_W = -q_\xi^{-1}J_W\frac{c}{J^2}$, so that we can write $\theta_{W,W} = \frac{c(cJ_W^2 q_{\xi,\xi}^{-1} - J^2 J_{W,W}q_\xi^{-1} + 2JJ_W^2 q_\xi^{-1})}{J^4}$. As the cost function and the value of the contract are positive, $\theta_{W,W} < 0$ if the numerator is negative, so if $cJ_W^2 q_{\xi,\xi}^{-1} - J^2 J_{W,W}q_\xi^{-1} + 2JJ_W^2 q_\xi^{-1} < 0$.

This is true if

$$\frac{q_{\xi,\xi}^{-1}}{q_\xi^{-1}} > \left(\frac{J_{W,W}}{J_W}\frac{J}{J_W} - 2\right)\frac{J}{c}.$$

As the right-hand side of the condition above is negative given the properties of J and $q_\xi^{-1} < 0$, the inequality would always be respected if $q_{\xi,\xi}^{-1} < 0$. We verify that the conditions holds for our matching function and parametrization.

The uniqueness of the optimum for the firm problem at object implies, given the properties of $\mathcal{V}$ and the theorem of the maximum, $W^*(y)$ is also continuous.

To show that $W_y^*(y) > 0$ note that, by the implicit function theorem, the derivative of **Equation A.17** with respect to y is:

$$(q_{W,W}J + 2q_W J_W + qJ_{W,W})W_y + q_W J_y + qJ_{W,y} = 0. \tag{A.22}$$

The first term in parenthesis is negative, as per the second order condition in **Equation A.20**. The second term is positive, given that J_y is positive (**Lemma A.3**) and q_W is positive as well. From **Equation A.15** it is possible to show that $J_{W,y} = \Gamma W_y$, with $\Gamma = \frac{u''}{(u')^2}\frac{\partial w}{\partial W} < 0$ as $\frac{\partial w}{\partial W} \geq 0$ and $u'' < 0$. Therefore,

$$\underbrace{(q_{W,W}J + 2q_W J_W + qJ_{W,W} + q\Gamma)}_{<0} W_y + q_W J_y = 0.$$

This means that, in order for the equality to be respected, $W_y^* > 0$.

□

Proposition A.4. *The mapping defined by the function $W : \mathcal{Y} \to \mathcal{V}$ is bijective for each worker characteristic (h, τ) and aggregate state Ω.*

Proof. As shown in **Lemma A.5**, the optimal promise for each entrepreneur with quality y is unique, continuous and monotonically increasing. Free entry implies that $\Pi(W^*(y)) = c(y)$, pinning down y in equilibrium. As $W^*(y)$ is unique and increasing, for each y there can be only one $W^*(y)$ so that the mapping between firm qualities and optimal promises is bijective. □

Corollary A.3. *The optimal search strategy of the workers and vacancy posting of the firms is increasing in aggregate productivity.*

Proof. We start from the free entry condition, ignoring without loss of generality τ, μ from the notation, and ignoring the indirect dependence of $q(\cdot)$ on $\theta(\cdot)$:

$$q(h, W; a)J(h, W, y; a) = c(y)$$

We know from **Lemma A.5** and **Proposition A.4** that the optimal W^* choice is a unique, monotonically increasing function $W(y)$ given workers characteristics (h, τ) and aggregate state Ω. Given **Proposition A.1**, as the worker search policy is continuous, unique and increasing in h, we can also conclude that firms and workers qualities feature positive assortative matching for any aggregate state, according to a continuous function $y = \xi(h; a)$. We thus want to find conditions for which $\xi_a > 0$, which would imply that in equilibrium the "assignment" (search) firm quality y for worker quality h is higher when the state of the economy a is better. We write ξ_a simply as $\frac{\partial y}{\partial a}$ in the proof.

We obtain the partial derivative of the free entry condition by y:

$$c'(y) = q\frac{\partial J}{\partial y} \tag{A.23}$$

We then write the total derivative of of this equation by a, ignoring terms featuring W by envelope condition:

$$q\left(\frac{\partial^2 J}{\partial y \partial a} + \frac{\partial^2 J}{\partial^2 y}\frac{\partial y}{\partial a}\right) + \frac{\partial J}{\partial y}\frac{\partial q}{\partial a} = c''(y)\frac{\partial y}{\partial a} \tag{A.24}$$

which, in turn, leads to:

$$\frac{\partial y}{\partial a} = \frac{q\frac{\partial^2 J}{\partial y \partial a} + \frac{\partial J}{\partial y}\frac{\partial q}{\partial a}}{c''(y) - \frac{\partial^2 J}{\partial^2 y}} \tag{A.25}$$

Given the fact that the sign of the denominator is negative given the properties of the function $c(\dot{)}$ and **Lemma A.4**, in order to sign the derivative it is sufficient to analyze the numerator.

$\frac{\partial y}{\partial a} > 0$ amounts to

$$q\frac{\partial^2 J}{\partial y \partial a} > -\frac{\partial J}{\partial y}\frac{\partial q}{\partial a}$$

We check for this condition in the solution and find it to be respected in the state space given our calibration.

□

Firm value $J(\cdot)$ is increasing in a, while retention is decreasing. As firm make more profits, they can marginally increase offered utilities to workers to maximize retention. The last inequality implies that for the workers to improve their target firm quality in better times it must be true that the first marginal effect, the increase in firm profits from better matches, must dominate the second, the relative decrease in retention, when aggregate productivity changes.

B Existence of a Block Recursive Equilibrium

In order to show that a Block Recursive Equilibrium (BRE) exists in our model we need to show that the equilibrium contracts, the workers' and the entrepreneurs value and policy functions do not depend on the distribution of employed and unemployed workers. This implies that the only element of the aggregate state that matters for a firm when making an hiring decision is the state of aggregate productivity but not the distribution of worker types (e.g. employed vs unemployed).

Proposition B.1. *A Block Recursive Equilibrium as defined in Definition 2.3 exists.*

Proof. We follow the approach in Menzio et al. (2016), Herkenhoff et al. (2023) and prove the existence of a BRE using backward induction.

Consider the lifetime values of an unemployed and an employed worker before the production stage in the last period of households lives with $\tau = T$:

$$U(h, T, \iota; \Omega) = u(b(h, T)) \tag{B.1}$$

$$V(h, T, \iota; \Omega) = u(w(a)), \tag{B.2}$$

their values trivially do not depend on the distribution of types as both valuations are 0 from $T + 1$ onward. Hence, $U(h, T, \iota; \Omega) = U(h, T, \iota; a)$ and $V(h, T, \iota; \Omega) = V(h, T, \iota; a)$.

The optimal contract for agents aged $\tau = T$, instead, solves the following problem

$$J(h, T, W, y; \Omega) = \sup_{w} [f(y, h; a) - w] \quad s.t.\; W = u(w),$$

that clearly does not depend on the distribution of worker types due to the directed search protocol and where the aggregate state only affects the promised utility and the optimal wage through realization of the aggregate productivity processes. Therefore, $J(h, T, \iota, W, y; \Omega) = J(h, T, \iota, W, y; a)$.

This also implies that the equilibrium market tightness

$$\theta(h, T, \iota, W; \Omega) = q^{-1}\left(\frac{c(y)}{J(h, T, \iota, W, y; a)}\right)$$

is independent from the distribution of worker types and it is only affected by realization of aggregate productivity, so $\theta(h, T, W; a)$.

This in turn implies that the search problem workers face at the beginning of the last period of their lives depends on the aggregate state only through aggregate productivity a:

$$R(h, T, \iota, V; a) = \sup_{v} \Big[p(\theta(h, T, \iota, v; a))\,[v - V]\Big],$$

does not depend on the distribution of worker types.

Stepping back at $\tau = T-1$, the value functions for the unemployed and the employed agents are solutions to the following dynamic programs

$$\sup_{v} u(b(h,T-1)) + \beta \mathbb{E}_{\Omega,\psi}\Big(U(h',T,\iota;a') + p(\theta(h,T,\iota,v;a'))\big[v - U(h',T,\iota,;a')\big]\Big)$$
$$u(w) + \beta \mathbb{E}_{\Omega,\psi}\left(\begin{array}{c} \lambda U(h',T,\iota;a') + \beta(1-\lambda)W + \\ +\beta(1-\lambda)\lambda_e \max(0, R(h',T,\iota,,W);a')]]\Big] \end{array}\right),$$

where both do not depend on the distribution of worker types.

The optimal contract at this step is a solution to

$$\begin{aligned} J_t(h,T-1,\iota,V,y;a) = \sup_{\{\pi_i,w_i,W_i\}} \sum_{i=1,2} \pi_i \Big(& f(y,h;a) - w_i \\ & + \mathbb{E}_{\Omega,\psi}\left[\widetilde{p}(h',T,W_{i,\Omega'};a')(J(h',T,y,W_i;a')\right]\Big) \end{aligned}$$

$$\begin{aligned} s.t.\ V = \sum_{i=1,2} \pi_i \left(u(w_i) + \mathbb{E}_{\Omega,\psi}\widetilde{r}(h',T,W_i;a')\right),\ & h' = \phi(h,y,\iota,\psi) \\ \mathbb{E}_{\Omega,\psi} \sum_{i=1,2} \pi_i \left(\mathbb{E}_{\Omega,\psi} J(h',T,\iota,W_i,y;a')\right) \geq 0 \text{ and } & t \leq T \end{aligned}$$

which does not depend on types distribution.

Therefore, also the equilibrium tightness and the search gain at $T-1$ are independent from types' distributions, as

$$\theta(h,T-1,\iota,W;a) = q^{-1}\left(\frac{c(y)}{J(h,T-1,\iota,W,y;a)}\right)$$
$$R(h,T-1,\iota,V;a) = \sup_{v}\Big[p(\theta(h,T-1,\iota,v;a))\big[v-V\big]\Big].$$

Stepping back from $\tau = T-1,...,1$ and repeating the arguments above completes the proof. $\square$

C Derivation of recursive contract SPFE

Solving the optimal contract and the overall model given the recursive structure obtained by following the promised utility method of Spear and Srivastava (1987) is computationally infeasible. This is due to the fact that the optimal contract requires to define a valid recursive domain and codomain of promised values that respects all the future forward looking constraints. Known solution methods for these kinds of models (Abreu, Pearce and Stacchetti, 1990), although robust, easily become computationally unmanageable as the number of states of the model increases. We thus follow Marcet and Marimon (2019) in deriving a recursive expression for the optimal contract in which the Lagrange multiplier for the promise keeping constraint **Equation A.15** is added as a co-state of the model, and allows us to circumvent the problem of searching for valid promised values domains altogether.

The reason why the recursive contracts method in Marcet and Marimon (2019) simplifies our problem is simple. As shown in **Equation A.15**, wage growth and levels in any next period and at every node are determined by the state-contingent multiplier on tomorrow's promise keeping constraints. This considerably reduces the complexity of the problem, as by definition Lagrange multipliers are defined over $\mathbb{R}^+$.

We follow Marcet and Marimon (2019) (hereby MM) and their terminology to define how a recursive saddle point functional equation (SPFE) can be obtained from the sequential formulation of the problem. For the present exposition of the constructive method to obtain the SPFE, for simplicity and without loos of generality, we ignore the randomization of the contract over the lotteries and the limited liability constraint. The latter choice, in particular, does not create any problem in terms of thinking about of developing the sequential problem over time: our choice of timing of exit decision is such as that exiting firms know form the start of their period whether the productivity level is below the critical one $a^*_{h,\tau,\iota,y}$ for the match (h, τ, ι, W, y), and thus whether they will exit or not. The lack of uncertainty and optimization over the next periods makes the problem of these firms, at some low states, equivalent to the problem of a firm with a lower maximum length (which is T, the retirement age, in general). At an exiting state t the firm knows *with certainty* that any $J_j = 0$ for $j > t$, match with a worker of age T.

Proposition C.1. *The solution to the contracting problem in* ***Equation*** (5) *is the same as the solution to the following saddle-point functional equation:*

$$\begin{aligned}\mathcal{P}_t(h_t, \tau_t, y_t, a_t, \gamma_t) =& \underset{\gamma_t}{inf}\, \underset{w_t}{sup}(f(a_t, y_t, h_t) - w_t) + \mu^1_t W_{y,t} - \gamma^1_t (W_{y,t} - u(w_t)) + \\ & \beta \mathbb{E}_t(\lambda U_{t+1} + (1-\lambda)\lambda_e p_{t+1} v^*_{t+1}) + \beta \mathbb{E}_t \widetilde{p}_{t+1} \mathcal{P}_{t+1}(h_{t+1}, \tau_{t+1}, y_{t+1}, a_{t+1}, \gamma_{t+1})\end{aligned}$$

with $\mu_t = \gamma_{t_1}$ *for some starting* γ_0.

Proof. Consider the problem[8]

[8] Without loss of generality, we ignore the level of education ι in the proof, as it is a fixed worker characteristic.

$$J_t(h_t,\tau_t,y_t,W_t,a_t)=\sup_{w_t,\{W_{s^{t+1}}\}}\Big(f(a_t,y_t,h_t)-w_t$$
$$+\mathbb{E}_{s^t}\left[\widetilde{p}(h_{t+1},\tau_{t+1},W_{s^{t+1}},a_{s^{t+1}})(J_{t+1}(h_{t+1},\tau_{t+1},y_t+1,W_{s^{t+1}},a_{s^{t+1}})\right]\Big) \tag{C.1}$$

$$\begin{aligned} s.t.\ W_t = u(w_t) + \beta\mathbb{E}_{s^t}\Big(&\lambda U_t(h_{t+1},\tau_{t+1},a_{t+1})+\\ &(1-\lambda)(\lambda_e p_{t+1}(h_{t+1},\tau_{t+1},W_{s^{t+1}},a_{s^{t+1}})v^*(h_{t+1},\tau_{t+1},W_{s^{t+1}},a_{s^{t+1}})\\ &+(1-\lambda_e p_{t+1}(h_{t+1},\tau_{t+1},W_{s^{t+1}},a_{s^{t+1}}))W_{s^{t+1}})\Big) \end{aligned} \tag{C.2}$$

We define as endogenous states $\mathbf{x}_t = [h_t,\tau_t,y_t,W_t]$, controls $\mathbf{c}_t = [w_t,W_{s^{t+1}}]\ \forall t, s^{t+1}$, whereas the only exogenous state is a_t. The endogenous states follow the law of motion

$$\mathbf{x}_{t+1} = \begin{bmatrix} h_{t+1} \\ \tau_{t+1} \\ y_{t+1} \\ W_{s^{t+1}} \end{bmatrix} = l(\mathbf{x}_t,\mathbf{c}_t,a_{s^{t+1}}) = \begin{bmatrix} \phi(h_t,y_t) \\ \tau_t+1 \\ y_t \\ W_{s^{t+1}} \end{bmatrix} \tag{C.3}$$

In the subsequent notation, where appropriate, we omit listing all states on which elements in the equation, and subsume their dependence under just listing the time t. J can be rewritten, by developing forward the recursion until time T, at which the match surely dissolves, as

$$J_t(\{h_t,\tau_t,y_t,W_t,a_t\}_{t=t_0}^{T-t_0}) = \mathbb{E}_{t_0}\sum_{t=t_0}^{T}\beta^{t-t_0}\prod_{i=0}^{t-t_0}\widetilde{p}_{t_o+i}\Big(f(a_t,y_t,h_t)-w_t\Big) \tag{C.4}$$

where $\widetilde{p}_{t_0} = 1$. Notice that the forward-looking constraint in **Equation C.2** is state contingent and an instance of it applies at *every* node of any possible history $s^t\ \forall t$ given the prevailing W_y promised at that node. The equilibrium is an instance of subgame perfect Nash equilibrium in which an agent chooses its strategies while anticipating the best response of the following agent, as common in dynamic games with a leader-follower component introduced by Von Stackelberg (1934). The structure of the problem and the solution also shares some commonality with Ramsey optimal policy problems in which a policy maker (in this case the firm) optimizes the utility of all agents according to some weights and taking into account their optimal behavior.[9]

[9] In the terminology of MM, we treat constraints coming from **Equation C.2** as a set of one period ahead forward looking constraint, which makes the analysis of our case akin to their case where one have $j=1$ forward looking constraints, and $N_1=0$. The difference with their problems, however, is that our problem features finite time, and thus each one period ahead forward looking constraint technically applies to a *different* function j_t (indexed by t).

We can redefine the problem:

$$V_{t_0}(\mathbf{x}_t, a_t) = \underset{\{w_{s^t}, W_{s^t}\}}{sup} \mathbb{E}_{t_0} \sum_{t=t_0}^{T} \beta^{t-t_0} \prod_{i=0}^{t-t_0} \widetilde{p}_{t_o+i} \Big(f(a_t, y_t, h_t) - w_t \Big) \tag{C.5}$$

$$s.t.\ [j=0]: \sum_{t=t_0}^{T} \beta^{t-t_0} \prod_{i=0}^{t-t_0} \widetilde{p}_{t_o+i} \Big(f(a_t, y_t, h_t) - w_t \Big) - R \geq 0 \tag{C.6}$$

$$[j=1, s^t]: \ W_{s^t} - u(w_{s^t}) - \beta \mathbb{E}_{s^t} \Big(\lambda U_{s^{t+1}} + (1-\lambda)(\lambda_e p_{s^{t+1}} v^*_{s^{t+1}}) + \widetilde{p}_{s^{t+1}} W_{s^{t+1}}) \Big) \geq 0 \tag{C.7}$$

where the constraint C.13 is a slack participation constraint for a sufficiently small R, so that the principal (the firm) is willing to enter the contract in the first place.

In the terminology of MM we can label

$$h_0^0(\mathbf{x}_t, \mathbf{c}_t, a_t) = f(a_t, y_t, h_t) - w_t \tag{C.8}$$

$$h_1^0(\mathbf{x}_t, \mathbf{c}_t, a_t) = f(a_t, y_t, h_t) - w_t - R \tag{C.9}$$

$$h_0^1(\mathbf{x}_t, \mathbf{c}_t, a_t) = W_t \tag{C.10}$$

$$h_1^1(\mathbf{x}_t, \mathbf{c}_t, a_t) = W_t - u(w_t) + \beta \mathbb{E}_t(\lambda U_{t+1} + (1-\lambda)\lambda_e p_{t+1} v^*_{y,t+1}) \tag{C.11}$$

and define the Pareto problem ($\mathbf{PP}_\mu$)

$$\mathbf{PP}_\mu : V_{\mu,t_0}(\mathbf{x}_t, a_t) = \underset{\{w_{s^t}, W_{s^t}\}}{sup} \mathbb{E}_{t_0} \sum_{t=t_0}^{T} \beta^{t-t_0} \prod_{i=0}^{t-t_0} \widetilde{p}_{t_0+i} \mu^0 \Big(f(a_t, y_t, h_t) - w_t \Big) + \mu^1 W_{t_0} \tag{C.12}$$

$$s.t.\ [j=0;\ \gamma^0]: \sum_{t=t_0}^{T} \beta^{t-t_0} \prod_{i=0}^{t-t_0} \widetilde{p}_{t_o+i} \Big(f(a_t, y_t, h_t) - w_t \Big) - R \geq 0 \tag{C.13}$$

$$[j=1, s^t;\ \gamma^1_{s^t}]: \ W_{s^t} - u(w_{s^t}) - \beta \mathbb{E}_{s^t} \Big(\lambda U_{s^{t+1}} + (1-\lambda)(\lambda_e p_{s^{t+1}} v^*_{s^{t+1}}) + \widetilde{p}_{s^{t+1}} W_{s^{t+1}}) \Big) \geq 0 \tag{C.14}$$

Still following the notation from Marcet and Marimon (2019), we can define the Saddle Point Problem ($\mathbf{SPP}_\mu$) as:

$$\mathbf{SPP}_{\mu} : SV_{\mu,t_0}(\mathbf{x}_{t_0}, a_{t_0}) = \underset{\{\gamma\in\mathbb{R}^l_+\}}{inf} \underset{\{w_{st}, W_{s t_0}\}}{sup} \mu^0\Big(f(a_{t_0}, y_{t_0}, h_{t_0}) - w_{t_0}\Big) + \mu^1 W_{t_0} +$$
$$+ \beta\mathbb{E}_t\left(\phi(\mu,\gamma)\sum_{i=0}^{T-t_0}\left[\beta^{t_0+i}\prod_{i=0}^{T-t_0-1}\widetilde{p}_{t_0+1+i}\left(f(a_{t_0+i}, y_{t_0+i}, h_{t_0+i}) - w_{t_0+i}\right) + W_{t_0+i}\right]\right) +$$
$$+ \gamma^1\left(u(w_{t_0} + \beta\mathbb{E}_{t_0}\left(\lambda U_{t_0+1} + (1-\lambda)\lambda_e p_{t_0+1} v^*_{y,t_0+1}\right)\right) +$$
$$+ \gamma^0\left(f(a_{t_0}, y_{t_0}, h_{t_0}) - w_{t_0} - R\right) \tag{C.15}$$

The problem can be restated as a saddle-point problem over a Lagrangian equation

$$\begin{aligned}
\underset{\gamma_t}{inf} \underset{\{w_{st}, W_{st}\}}{sup} \; & \mu^0\Big(f(a_{t_0}, y_{t_0}, h_{t_0}) - w_{t_0}\Big) + \mu^1 W_{t_0} + \\
& \gamma^0\Big((f(a_{t_0}, y_{t_0}, h_{t_0}) - w_{t_0}\Big) - R) + \\
& \gamma^1_{t_0}\Big(-W_{t_0} + u(w_{t_0}) + \beta\mathbb{E}_{t_0}(\lambda U_{t_0+1} + (1-\lambda)(\lambda_e p_{t_0+1} v^*_{t_0+1} + \widetilde{p}_{t_0+1} W_{t_0+1})\Big) + \\
& \beta\mathbb{E}_{t_0}\Bigg[(\mu_0 + \gamma_0)\sum_{t=t_0+1}^{T}\beta^{t-t_0-1}\prod_{i=0}^{T-t_0-1}\widetilde{p}_{t_0+1+i}\Big(f(a_t, y_t, h_t) - w_t\Big) + \\
& \sum_{t=t_0+1}^{T}\mathbb{E}_t\beta^{t-t_0-1}\prod_{i=0}^{t-t_0-1}\widetilde{p}_{t_0+1+i}\gamma^1_t\Big(-W_t + u(w_t) + \\
& \beta(\lambda U_{t+1} + (1-\lambda)(\lambda_e p_{t+1} v^*_{t+1} + \widetilde{p}_{t+1} W_{t+1})\Big)\Bigg]
\end{aligned} \tag{C.16}$$

which, thanks to some algebra and the law of iterated expectations becomes

$$\begin{aligned}
\underset{\gamma_t}{inf} \underset{\{w_{st}, W_{st}\}}{sup} - \gamma^0 R + \mathbb{E}_{t_0}\sum_{t=t_0}^{T}\beta^{t-t_0}\prod_{i=0}^{t-t_0}\widetilde{p}_{t_0+i}\Bigg[&\Big(\mu^0_t + \gamma^0_t\Big)\Big(f(a_t, y_t, h_t) - w_t\Big) + \mu^1_t W_t - \\
&\gamma^1_t\Big(W_t - u(w_t) - \beta(\lambda U_{t+1} - (1-\lambda)\lambda_e p_{t+1} v^*_{t+1}\Big)\Bigg]
\end{aligned} \tag{C.17}$$

where $\mu^0_t = \mu^0 = 1$, $\gamma^0_t = \gamma_0 = 0$, $\mu^1_t = \gamma^1_{t-1}$ for some starting $\gamma^1_{t_0-1}$.

The problem can now be written in recursive form. Define

$$\mathcal{P}_t(h_t, \tau_t, y_t, a_t, \gamma_t) = \underset{W_t}{sup}\, J_t(h_t, \tau_t, y_t, W_t, a_t) + \mu^1_t W_t \tag{C.18}$$

Given **Equation C.17** the SPFE of the problem can be written as

$$\mathcal{P}_t(h_t, \tau_t, y_t, a_t, \gamma_t) = \underset{\gamma_t}{inf} \underset{w_t}{sup} (f(a_t, y_t, h_t) - w_t) + \mu_t^1 W_t - \gamma_t(W_t - u(w_t)) + \beta \mathbb{E}_t(\lambda U_{t+1} + (1-\lambda)\lambda_e p_{t+1} v_{t+1}^*) + \beta \mathbb{E}_t \widetilde{p}_{t+1} \mathcal{P}_{t+1}(h_{t+1}, \tau_{t+1}, y_{t+1}, a_{t+1}, \gamma_{t+1}) \tag{C.19}$$

One can easily verify that the solution of this equation is the same we found in the maximization of **Equation 5** in the main text. Take the first order conditions and compute the envelope condition:

$$[FOC\ w_t]: \ -1 + \gamma_t u'(w_t) = 0 \tag{C.20}$$

$$[ENV\ W_t]: \ \frac{\partial \mathcal{P}_t}{\partial W_t} = \mu_t^1 - \gamma_t \tag{C.21}$$

$$[FOC\ W_{t+1}]: \ -\widetilde{p}_{t+1} W_{t+1} \gamma_t + \frac{\partial \widetilde{p}_{t+1}}{\partial W_{t+1}} \mathcal{P}_{t+1} + \widetilde{p}_{t+1} \frac{\partial \mathcal{P}_{t+1}}{\partial W_{t+1}} = 0 \tag{C.22}$$

where **EquationC.22** is obtained by adding and subtracting from **Equation C.19** $\beta \gamma_t \widetilde{p}_{t+1} W_{t+1}$. The reader should also keep in mind that the condition in **Equation C.22** is actually state contingent and applied to *all* future states next period, with a different set of co-states $\gamma_{s^{t+1}}$ for each realization of a_{t+1}.

Some rearranging of the **Equation C.22** leads to the following result

$$\frac{\partial \log \widetilde{p}_{t+1}}{\partial W_{t+1}} \Big(\mathcal{P}_{t+1} - \gamma_t W_{t+1} \Big) = \gamma_{t+1} - \mu_{t+1}^1 \tag{C.23}$$

which, given the law of motion of the co-states and the definition in **Equation C.18** can be re-written as:

$$\frac{\partial \log \widetilde{p}_{t+1}}{\partial W_{t+1}} J_{t+1} = \frac{1}{u'(w_{t+1})} - \frac{1}{u'(w_t)} \tag{C.24}$$

which is exactly **Equation 14**, namely the Euler equation that governs the behavior of wage setting and disciplines the provision of insurance within the contract. □

D Data construction and sample selection

We rely on two main data sources provided by INPS through the VisitINPS Program: i) data on employment relationships in Italy from 1996 to 2018 (*Uniemens* dataset) and ii) balance sheet data for incorporated Italian firms from 1996 to 2018 (*Cerved* dataset).

Starting from data on virtually the universe of Italian private employment relationships at the contract level, we calculated for each worker their yearly gross real earnings and for the workers with multiple contracts we selected the information associated to the highest paid contract in the year and we find the associated annualized income using information on the number of actual weeks worked. For workers that experience a job flow or are promoted after 2009 we obtain the education level from the *Comunicazioni Obbligatorie* dataset, provided to INPS by the Ministry of Labor, and identify graduates and non-graduates. We label as graduates in the data both workers getting undergraduate education or above, or workers getting specialized diplomas after high-school.

We restrict our focus on workers employed under either full-time or part-time working schemes between 16 and 65 years old. Moreover, in order to compute AKM fixed effects and maintain the estimation computationally feasible, we analyze the connected set of firms and workers across firms with more than 15 employees. As in Bonhomme, Lamadon and Manresa (2019) we first cluster firms by means of a weighted K-means clustering based on deciles of their wage distribution. The weight for the clustering is the yearly number of employees.

From *Cerved*, we have access to firm balance sheet data. From this dataset we obtain information of value added, overall labor costs and calculate quantiles of value added per employee. All monetary values (wages, value added, total compensation) are trimmed at the 1% level and deflated by the consumer price index for Italy.

E Model solution and estimation

Model solution. For each education level, we solve the model by backward induction on a regular grid of human capital, wage multiplier levels, firm quality and aggregate shocks. In particular, starting from the last period of workers lives we compute the terminal wage in each state, then solve the search problems for both the employed and the unemployed, which allows us to compute the market tightness in each submarket as well as the value of the contract and of unemployment. With these objects in hand we proceed backwards until workers labor market entry.

In the model simulations, we populate each cohort of agents with 42 individuals, 30 non-graduates and 12 graduates. Graduates enter the labor market in a staggered manner every quarter for the first three years of their life. We then simulate the model for 600 periods (we take 180 periods as burn in) and construct a panel with 419 periods and $7,560$ individuals per period. In the model, we consider one period as one quarter in the data.[10]

[10] We use a grid of $20 \times 15 \times 25 \times 7$ for each education level and we consider 180 quarters of workers life. Solving and simulating the model takes approximately 25 minutes with Python 3.11 on a Linux

Estimation. To pin down the 18 internal parameters in the model, $\boldsymbol{\theta}$, we minimize the absolute error between model simulated moments, $m(\boldsymbol{\theta})$, and their empirical counterparts, $\mathbf{d}$. More formally, we pick the vector of parameters

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \left\{ (|m(\boldsymbol{\theta}) - \mathbf{d}|)' W (|m(\boldsymbol{\theta}) - \mathbf{d}|) \right\}.$$

In $\mathbf{d}$ and $m(\boldsymbol{\theta})$ we stack: the profiles of E-E and E-U transitions by age (9 age groups); wage growth relative to the initial wage by education level and experience (9 experience groups); the unemployment rate; the correlations of labor market flows with the cyclical component of aggregate output; the average inactivity rate; average sorting, measured as the average correlation between AKM fixed effects in the data and as the correlation of worker and firm qualities in the model; the ratio of initial wages between graduates and non-graduates, and the ratios of the second and third quintile of value added per employee to the first.

Jointly, these ten moments deliver 44 restrictions for 18 parameters. We construct each moment in the model weighting by age from Italian population weights. In the SMM we weight moments so that each age profile, as a whole, has the same weight of the other moments and, within each profile, we give more weight to age groups that have a higher demographic share in the Italian population.[11] We collect this weighting scheme in the matrix W.

We solve for $\boldsymbol{\theta}^*$ numerically adopting two complementary strategies. In the first, we use a two step procedure and minimize the distance between model-generated and data-generated moments using a global solver on the largest feasible domain, for this, we rely on the particle swarm algorithm contained in the library developed by (Blank and Deb, 2020). In the second, we solve and simulate the model over a sparse grid of the parameter space.[12] In both cases, we refine the estimation using the best combination of parameters as the starting point of a local minimization routine, for this we rely on the Nelder-Mead algorithm in the SciPy library. **Table 3** in the paper reports the averages for the profiles and the other targets, **Figure E.1** plots the whole profiles.

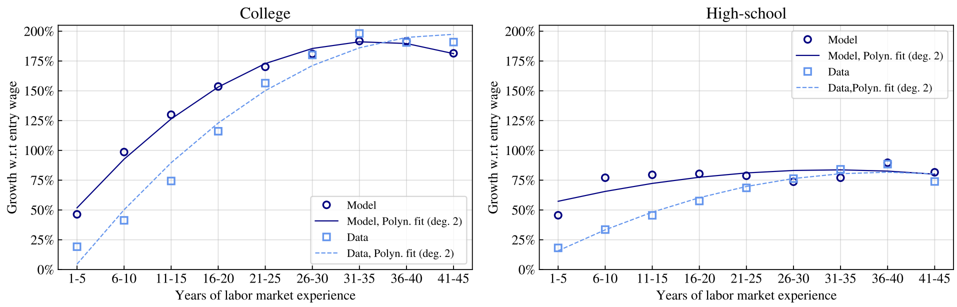
HPC cluster with 52 cores and 120G of RAM.

[11]For example, the weight for the EE rate of 18-22 years old has a weight equal to $1/10 \times 0.091$, the share of 18-22 years old in the Italian population, while the weight on the unemployment rate instead is 1/10, so that the entire EE profile, as a whole, weights as much as the unemployment rate.

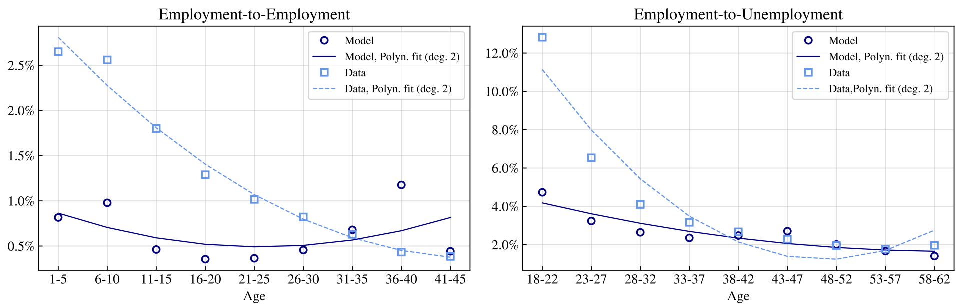
[12]The advantage of selecting the parameters with this procedure is that the grid is built so that the function domain is optimally covered with the least amount of possible points compared to other forms of approximation (e.g. equi-spaced or random grids). The sparse grid is built using the"Tasmanian" libraries (Stoyanov, 2015).

Figure E.1. Target profiles

(a) Wage Profiles



(b) Transition rates



F Additional Figures and Tables

In order to estimate the effect of entering the labor market in a recession we use an age-cohort-period model in which we break the collinearity among the three set of fixed effects by proxying the cohort fixed effects with the cyclical component of real GDP (Hamilton filtered). In particular, we estimate the following yearly model:

$$\log(w)_{t,c,e} = \Phi_t + \Phi_e + \beta_e \widetilde{Y}_c \times \Phi_e + u_{t,c,e}, \tag{F.1}$$

where Φ_t, Φ_e, are dummies for calendar years and labor market experience, and $\widetilde{Y}_c$ is the cyclical realization of real GDP for cohort c at time of their labor market entry. The set of coefficients β_e, therefore, estimate the effect of aggregate conditions on real wages at each year of labor market experience.

Table F.1. Effects of initial aggregate conditions along the experience profile and experience growth profile

Dep.Variable: Log-Wage	Experience × Cycle	Experience
Experience Dummy		
0	1.968	
	(0.452)	
1	1.463	0.150
	(0.238)	(0.011)
2	1.473	0.246
	(0.283)	(0.014)
3	1.239	0.304
	(0.343)	(0.014)
4	1.250	0.342
	(0.349)	(0.015)
5	1.239	0.375
	(0.349)	(0.015)
6	1.301	0.399
	(0.287)	(0.015)
Age FE	✓	✓
Year FE	✓	✓
Sex FE	✓	✓
LLM FE	✓	✓
R^2	0.89	0.89
N	254,000,000	254,000,000

Note: The table reports regression coefficients for the empirical estimates in the data used to construct the profiles in Figure 8.

State dependence. We check for state dependence by running the following regression across different model simulations

$$Y_{\text{Post}} = \alpha + \textstyle\sum_{j=1}^{4} \beta_j \underbrace{\mathbb{E}[FQ^j_{\text{Pre}}]}_{\text{Firm Quality}} + \gamma_j \overbrace{\mathbb{E}[HC^j_{\text{Pre}}]}^{\text{Human Capital}} + \delta_j \underbrace{\mathbb{E}[W^j_{\text{Pre}}]}_{\text{Wages}} + \eta_j \overbrace{\mathbb{E}[LS^j_{\text{Pre}}]}^{\text{Labor Share}} + \theta \underbrace{\mathbb{C}(FQ_{\text{Pre}}, HC_{\text{Pre}})}_{\text{Sorting}} + \varepsilon,$$

in which $\mathbb{E}[X^j_{\text{Pre}}]$, denotes the j^{th} moments of an endogenous, cross-sectional distribution before the shock, and Y_{Post} are model outcomes after the shock.

Cyclical behaviour of the labor share. We check the cyclical behaviour of the labor share, conditional on TFP shocks both in the model and in the data. In the data, we estimate a quarterly bi-variate VAR with $\mathbf{y}_t = [\texttt{Labor Share}'_t, \texttt{Real GDP}'_t]'$. We

Table F.2. State Dependence

(a) Mean

	Output Response		Persistence
	Short-term	Medium-term	
Firm Quality	**-1.6***** (0.6)	**-2.1**** (1.0)	-0.5 (1.0)
Human Capital	**2.1**** (0.8)	1.6 (1.4)	**2.6*** (1.4)
Wage	0.1 (0.4)	0.3 (0.6)	-0.0 (0.6)
Labor Share	**-2.4***** (0.6)	**-4.1***** (1.1)	**1.8*** (1.1)
Sorting	**2.1**** (1.0)	**3.4**** (1.7)	0.9 (1.6)
R^2	0.6	0.5	0.1
N	100	100	100

(b) Variance

	Output Response		Persistence
	Short-term	Medium-term	
Firm Quality	**0.2*** (0.1)	**0.3*** (0.2)	0.1 (0.2)
Human Capital	**-1.1*** (0.5)	**-1.4*** (0.8)	-1.2 (0.8)
Wage	-0.2 (0.2)	-0.3 (0.4)	0.0 (0.4)
Labor Share	-10.2 (9.2)	-6.9 (15.8)	-18.6 (15.3)
R^2	0.6	0.5	0.1
N	100	100	100

(c) Skewness

	Output Response		Persistence
	Short-term	Medium-term	
Firm Quality	**-0.6**** (0.3)	**-0.9*** (0.5)	-0.2 (0.5)
Human Capital	1.2 (0.9)	0.3 (1.6)	**2.6*** (1.5)
Wage	**-0.4**** (0.2)	-0.3 (0.3)	**-0.7**** (0.3)
Labor Share	0.3 (0.3)	0.3 (0.5)	**0.9*** (0.5)
R^2	0.6	0.5	0.1
N	100	100	100

(d) Kurtosis

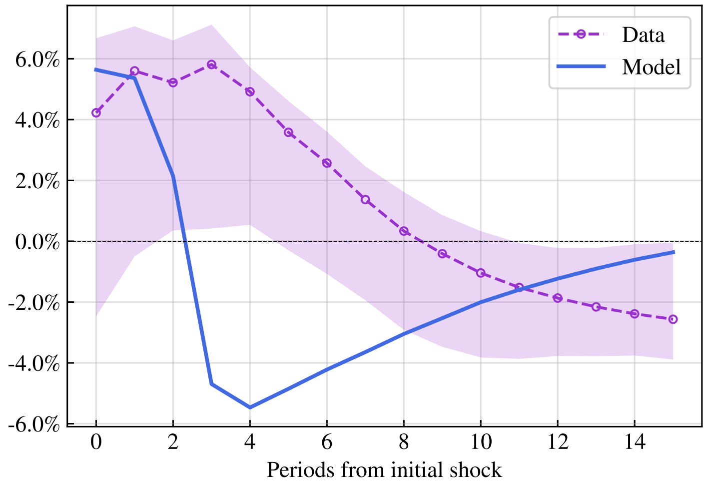
	Output Response		Persistence
	Short-term	Medium-term	
Firm Quality	**0.05**** (0.03)	**0.1*** (0.04)	0.0 (0.0)
Human Capital	-0.2 (0.1)	-0.1 (0.2)	-0.4 (0.2)
Wage	0.0 (0.0)	0.0 (0.1)	**0.1**** (0.1)
Labor Share	0.2 (0.2)	0.2 (0.4)	-0.1 (0.4)
R^2	0.6	0.5	0.1
N	100	100	100

Note: Standard-errors in parenthesis. p-value: * < 0.1, ** < 0.05, *** < 0.01. The table reports the coefficients of regressing model outcomes after the shock on moments of relevant endogenous variables before the shock (one year average). The results are scaled to report the effect of changing the regressors by 1pp. Sorting is computed every quarter as the correlation between firm and worker quality and reported in the same table of the means even if technically is not a moment of a distribution. Model outcomes are: i) short-term cumulative output response (1 year after the shock); ii) the medium-term cumulative output response (3 years after the shock); and iii) the persistence of the shock (number of quarters before the output IRF is back at zero). For ease of exposition, the results are grouped by moments but the coefficients are computed including all moments in the same regression. The simulations are those underlying **Figure 10**.

quadratically detrend the variables and include four lags for each variables[13] and we identify productivity shocks using long-run restrictions. **Figure F.1** reports the IRFs to a TFP shocks of the same magnitude in the model and in the data during the recovery.

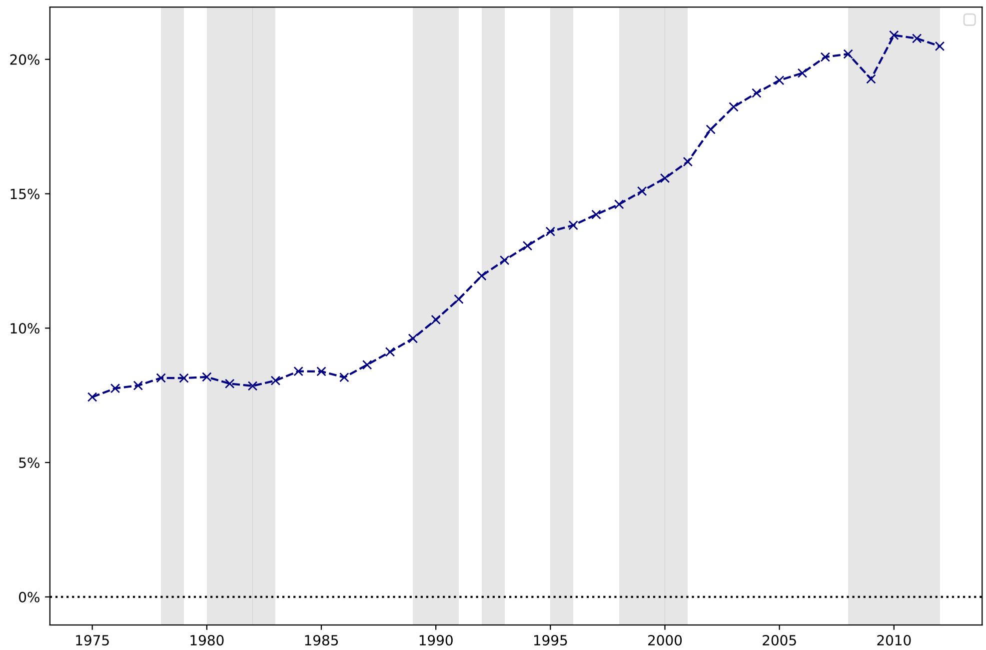
[13]We select the number of lags with the number that minimized AIC information criterion.

Figure F.1. Labor Share Dynamics After a Recessionary Shock



Note: The figure reports the IRFs after a TFP shock in the VAR and in the model. Shaded area is the 95% confidence interval from 500 bootstrap replications. *Data source*: FRED and ISTAT.

Figure F.2. Tertiary School Enrollment and the Business Cycle



Note: The figure plots the ratio of students enrolled in tertiary education to population aged 16-29 in every year. Source: Italian National Institute of Statistics (ISTAT). Shaded areas indicate the OECD based Recession Indicators for Italy.

Table F.3. E-U-E transitions

(a) Data

Dep.Variable: Log-wage after E-U-E transition	(1)	(2)
Quality of origin firm (FQ):		
2^{nd} quint.	0.100	0.128
	(0.002)	(0.004)
3^{rd} quint.	0.143	0.210
	(0.002)	(0.004)
4^{th} quint.	0.153	0.182
	(0.002)	(0.004)
5^{th} quint.	0.230	0.260
	(0.002)	(0.004)
Log-wage at origin	0.669	0.609
	(0.001)	(0.002)
Experience controls	✓	✓
Sex FE	✓	✓
Year FE	✓	✓
Contract type FE	✓	✓
Full- & Part-Time FE	✓	✓
Justified dismissals	✓	
R^2	0.48	0.36
N	955,602	338,975

(b) Model

Dep. Variable: Log-wage after E-U-E transition	(1)
Firm quality bracket	
2 of 3	0.104***
	(0.025)
3 of 3	0.171***
	(0.034)
Log-wage at origin	0.075***
	(0.009)
Controls	✓
Observations	19,231
R^2	0.106

*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Note: Standard errors in parentheses. The tables report a specification on datasets based on workers that experience experience an Employment to Unemployment to Employment transitions (E-U-E). In Panel **(a)**, column (2) excludes separations that are justified in the Italian labor law (*giusta causa*). In the model, *Controls* include the pre-transition human capital and a polynomial in labor market experience. Referenced on page(s) [23] .

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